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Annealed n-Vector p-Spin Model

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A disordered n-vector model with p spin interactions is introduced and studied in mean field theory for the annealed case. We present complete solutions for the cases $n=2$ and $n=3$, and have obtained explicit order parameter equations for all the stable solutions for arbitrary n. For all n and p we find one stable high-temperature phase and one stable low-temperature phase. The phase transition is of first order. For $n=2$, it is continuous in the order parameters for $p \leqslant 4$ and has a jump discontinuity in the order parameters if $p>4$. For $n=3$, it has a jump discontinuity in the order parameters for all p.

KEY WORDS: Disordered spin systems; n-vector model; mean field theory.

1. INTRODUCTION

In 1968 Stanley[1] introduced the n-vector model as a unifying description of many simpler nonrandom models in statistical mechanics such as the Ising model ($n=1$), the Vaks–Larkin plane rotator model ($n=2$), the classical Heisenberg model ($n=3$), and the Berlin–Kac spherical model ($n=\infty$).

Stanley's exact solutions have been confined to nearest neighbor one-dimensional chains and hence do not exhibit a phase transition. The mean field theory obtained by considering this model with an infinite-range potential (and hence a phase transition) was studied by Silver *et al.*[2]

We have generalized the n-vector model by introducing Gaussian random bonds and p spin interactions. In this paper we shall be considering the mean field theory for the annealed case. The quenched case is intended for a followup publication.

For random spin systems, even mean field theory has proven to be very subtle. The first infinite-range Ising spin glass model was proposed by

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Sherrington and Kirkpatrick (SK). In 1980 Derrida[3] showed that the SK model could be generalized to models involving p spin interactions and that in the limit of $p \to \infty$ they simplified to a random energy model, which consists of a collection of independently distributed random energy levels. He was then able to solve this model without recourse to the n-replica trick. Gross and Mézard[4] confirmed his results for the same $p \to \infty$ model by using the n-replica method and Parisi's replica-symmetry-breaking scheme. Gardner[5] and Stariolo[6] have studied the model for finite p. They find that for $p = 2$ and $p = \infty$ there are two phases, a high-temperature phase above a critical temperature T_c and a spin-glass phase below T_c. The phase transition is of second order and continuous in the order parameter $q(x)$ for $p = 2$ but has a jump discontinuity in the order parameter for $p = \infty$. For all $p > 2$ there are three phases, a high-temperature phase above a critical temperature T_{c_1}, a spin-glass phase SG1 which is stable between T_{c_1} and a second critical temperature $T_{c_2} < T_{c_1}$, and a spin-glass phase SG2 below T_{c_2}. The phase transition at T_{c_1} is of second order with no latent heat but displays a jump discontinuity in the order parameter. The phase transition at T_{c_2} is of second order and continuous in the order parameter. Although a stability analysis shows that the disordered high-temperature solution is stable at all temperatures, its entropy becomes negative at some temperature $T' < T_{c_1}$. This suggests that replica symmetry is broken. By performing the first step in Parisi's replica-symmetry-breaking scheme one obtains the spin-glass phase SG1. The nature of the spin-glass phase SG2, however, is not completely understood, since the full replica-symmetry-breaking scheme would have to be performed in this case.

Studying a solvable random spin model with p spin interactions must therefore be of some value. It turns out that already in the annealed case our n-vector model with p spin interactions displays a considerable richness of solutions and subtleties regarding their stability. These annealed solutions will further constitute the basis for the quenching of the model by means of the n-replica trick.

The paper is organized as follows. In Section 2 we present our model and derive the order parameter equations by means of a saddle point method and a theorem from the theory of matrices. In Section 3 we present the complete solution for the $n = 2$ model and in Section 4 we present it for the $n = 3$ model. We find that out of all possible solutions for the order parameter matrix Q (defined in the next section), only the diagonal solutions with at most two distinct eigenvalues are stable. By extrapolating this result to the case of general n, we then derive in Section 5 explicit forms of the order parameter equations for all stable solutions.

2. MODEL AND ORDER PARAMETER EQUATIONS

The model studied is defined by a generalized Hamiltonian

$$-\beta\mathscr{H} \equiv \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant N} J_{i_1 \cdots i_p} \sum_{\alpha=1}^{n} S_{i_1}^{\alpha} \cdot \cdots \cdot S_{i_p}^{\alpha} \tag{2.1}$$

where $\mathbf{S}_i$ is an n-vector spin $\mathbf{S}_i \equiv (S_i^1, S_i^2,..., S_i^n)$, normalized to $\|\mathbf{S}_i\| = \sqrt{n}$. The coupling constants $J_{i_1 \cdots i_p}$ are independent random variables with an appropriately scaled Gaussian distribution so as to give rise to an intensive free energy per spin

$$P(J_{i_1 \cdots i_p}) \equiv \left[\frac{N^{p-1}}{\pi p!\,(\Delta J)^2}\right]^{1/2} \exp\left[-\frac{(J_{i_1 \cdots i_p})^2\, N^{p-1}}{p!\,(\Delta J)^2}\right], \qquad \Delta J \equiv \beta\, \Delta\tilde{J} \tag{2.2}$$

$\Delta\tilde{J}$ represents the width of the Gaussian distribution, which for simplicity is assumed to be centered at $J_0 = 0$. The case of a nonzero mean can be treated in a canonical fashion.

For $n = 1$ our model represents the random Curie–Weiss model with p spin interactions. For $n = 2$ we obtain the random planar rotator model with p spin interactions. For $n = 3$ we have the random classical Heisenberg model with p spin interactions. The case $p = 2$ is the random Stanley model. All of these models have well-known submodels, such as the Sherrington–Kirkpatrick model for $n = 1$ and $p = 2$ and the random energy model for $n = 1$ and $p \to \infty$. However, we do not recover the random spherical model for $p = 2$ and $n \to \infty$, since this would require $n = N$ and hence a different limiting procedure and scaling.

From Eqs. (2.1) and (2.2) we form the annealed partition function

$$\langle Z_N \rangle = \int_{-\infty}^{\infty} \prod_{1 \leqslant i_1 < \cdots < i_p \leqslant N} P(J_{i_1 \cdots i_p})\, dJ_{i_1 \cdots i_p} \times \mathrm{Tr}_{\{\mathbf{S}_i\}} \exp\left[\sum_{\alpha=1}^{n} \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant N} J_{i_1 \cdots i_p} S_{i_1}^{\alpha} \cdot \cdots \cdot S_{i_p}^{\alpha}\right] \tag{2.3}$$

Evaluating the Gaussian integral gives

$$\begin{aligned}\langle Z_N \rangle &= \mathrm{Tr}_{\{\mathbf{S}_i\}} \exp\left[\frac{p!\,(\Delta J)^2}{4N^{p-1}} \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant N} \left(\sum_{\alpha=1}^{n} S_{i_1}^{\alpha} \cdot \cdots \cdot S_{i_p}^{\alpha}\right)^2\right] \\ &= \mathrm{Tr}_{\{\mathbf{S}_i\}} \exp\left[\frac{p!\,(\Delta J)^2}{4N^{p-1}} \sum_{\alpha,\beta=1}^{n} \sum_{1 \leqslant i_1 < \cdots < i_p \leqslant N} S_{i_1}^{\alpha} S_{i_1}^{\beta} \cdot \cdots \cdot S_{i_p}^{\alpha} S_{i_p}^{\beta}\right] \\ &= \mathrm{Tr}_{\{\mathbf{S}_i\}} \exp\left\{\frac{(\Delta J)^2}{4N^{p-1}}\left[N^p \sum_{\alpha,\beta=1}^{n} q_{\alpha\beta}^p + O(N^{p-1})\right]\right\}\end{aligned} \tag{2.4}$$

where we have defined

$$q_{\alpha\beta} \equiv \frac{1}{N} \sum_{i=1}^{N} S_i^\alpha S_i^\beta = O(1) \qquad \text{as} \quad N \to \infty \tag{2.5}$$

We evaluate the trace in Eq. (2.4) by introducing a Lagrange multiplier matrix $\lambda_{\alpha\beta}$. In the limit of large N, $\langle Z_N \rangle$ becomes

$$\langle Z_N \rangle \xrightarrow{N \to \infty} \int_{-\infty}^{\infty} \prod_{\alpha,\beta} dq_{\alpha\beta} \int_{-i\infty}^{i\infty} \prod_{\alpha,\beta} \frac{d\lambda_{\alpha\beta}}{2\pi} \exp[NG(q_{\alpha\beta}, \lambda_{\alpha\beta})] \left(\frac{N}{2}\right)^{n^2} \tag{2.6}$$

where

$$G(q_{\alpha\beta}, \lambda_{\alpha\beta}) \equiv \frac{(\Delta J)^2}{4} \sum_{\alpha,\beta} q_{\alpha\beta}^p - \frac{1}{2} \sum_{\alpha,\beta} \lambda_{\alpha\beta} q_{\alpha\beta} + \ln \operatorname{Tr}_{\{S\}} \exp\left(\frac{1}{2} \sum_{\alpha,\beta=1}^{n} \lambda_{\alpha\beta} S^\alpha S^\beta\right) \tag{2.7}$$

Equation (2.6) can then be evaluated by the method of steepest descent

$$\langle Z_N \rangle \xrightarrow{N \to \infty} \exp[NG^*] \cdot C \tag{2.8}$$

where C is a constant independent of N and where G^* is the dominant saddle point of G.

From now on, Q denotes the $n \times n$ matrix with elements $q_{\alpha\beta}$ and $Q^{(k)}$ denotes the $n \times n$ matrix with elements $q_{\alpha\beta}^k$. Similarly, Λ ($\Lambda^{(k)}$) denotes the $n \times n$ matrix with elements $\lambda_{\alpha\beta}$ ($\lambda_{\alpha\beta}^k$). The matrices Λ and Q are defined by the saddle point equations

$$\frac{\partial G}{\partial \lambda_{\alpha\beta}} = 0, \qquad \frac{\partial G}{\partial q_{\alpha\beta}} = 0 \tag{2.9}$$

Evaluating these equations yields

$$\lambda_{\alpha\beta} = \frac{p(\Delta J)^2}{2} q_{\alpha\beta}^{p-1} \tag{2.10}$$

$$q_{\alpha\beta} = \frac{\int_{\|\mathbf{S}\| = \sqrt{n}} S^\alpha S^\beta \exp[\frac{1}{2}\mathbf{S}^T \Lambda \mathbf{S}]\, d\mathbf{S}}{\int_{\|\mathbf{S}\| = \sqrt{n}} \exp[\frac{1}{2}\mathbf{S}^T \Lambda \mathbf{S}]\, d\mathbf{S}} \tag{2.11}$$

where $\mathbf{S}^T$ is the transposed vector $\mathbf{S}$. By using Eq. (2.10), one can show that for odd p only solutions with $q_{\alpha\beta} \geqslant 0$ can constitute saddle points of G. This is a manifestation of the fact that for odd p and in the limit of large N only states with $q_{\alpha\beta} \geqslant 0$ will contribute to the trace in Eq. (2.4). Thus, we have the condition

$$q_{\alpha\beta} \geqslant 0 \qquad \text{if} \quad p = \text{odd} \tag{2.12}$$

imposed by our model on the order parameters $q_{\alpha\beta}$. Inserting Eq. (2.10) into Eq. (2.11) finally gives the following equation for the order parameter matrix Q:

$$Q = \frac{\int_{\|\mathbf{S}\| = \sqrt{n}} \mathbf{S}\mathbf{S}^T \exp[\frac{1}{4} p(\Delta J)^2 \, \mathbf{S}^T Q^{(p-1)} \mathbf{S}] \, d\mathbf{S}}{\int_{\|\mathbf{S}\| = \sqrt{n}} \exp[\frac{1}{4} p(\Delta J)^2 \, \mathbf{S}^T Q^{(p-1)} \mathbf{S}] \, d\mathbf{S}} \tag{2.13}$$

We are now going to formally evaluate this order parameter equation. Because of Eq. (2.5), the order parameter matrix Q must be symmetric. Thus, there exists an orthonormal coordinate transformation

$$\mathbf{S} = O\hat{\mathbf{S}} \tag{2.14}$$

which diagonalizes $Q^{(p-1)}$,

$$O^T Q^{(p-1)} O = \text{diag}(\lambda_1, \ldots, \lambda_n) \tag{2.15}$$

Here we have denoted by $\lambda_1, \ldots, \lambda_n$ the eigenvalues of $Q^{(p-1)}$ and by O a suitable orthonormal matrix.

Any orthonormal coordinate transformation will map the n-sphere

$$\mathcal{S}_n = \{\mathbf{S} \mid \|\mathbf{S}\| = \sqrt{n}\} \tag{2.16}$$

onto itself. If we make the coordinate transformation (2.14) in Eq. (2.13), we therefore get

$$O^T Q O = \frac{\int_{\|\hat{\mathbf{S}}\| = \sqrt{n}} \hat{\mathbf{S}}\hat{\mathbf{S}}^T \exp[\frac{1}{4} p(\Delta J)^2 \sum_{\gamma=1}^{n} \lambda_\gamma (\hat{S}^\gamma)^2] \, d\hat{\mathbf{S}}}{\int_{\|\hat{\mathbf{S}}\| = \sqrt{n}} \exp[\frac{1}{4} p(\Delta J)^2 \sum_{\gamma=1}^{n} \lambda_\gamma (\hat{S}^\gamma)^2] \, d\hat{\mathbf{S}}} \tag{2.17}$$

where we have also used the fact that $O^T O = 1$.

Let us now define the functions

$$g(\sqrt{n}) \equiv \int_{\|\hat{\mathbf{S}}\| = \sqrt{n}} \exp\left[\frac{p(\Delta J)^2}{4} \sum_{\gamma=1}^{n} \lambda_\gamma (\hat{S}^\gamma)^2\right] d\hat{\mathbf{S}} \tag{2.18}$$

$$f_{\alpha\beta}(\sqrt{n}) \equiv \int_{\|\hat{\mathbf{S}}\| = \sqrt{n}} \hat{S}^\alpha \hat{S}^\beta \exp\left[\frac{p(\Delta J)^2}{4} \sum_{\gamma=1}^{n} \lambda_\gamma (\hat{S}^\gamma)^2\right] d\hat{\mathbf{S}} \tag{2.19}$$

In order to avoid the $\sqrt{n}$ constraint in these integrals, we form the Laplace transform

$$\int_0^\infty \exp[-\lambda x] \frac{f_{\alpha\beta}(\sqrt{x})}{2\sqrt{x}} \, dx$$
$$= \int_0^\infty \exp[-\lambda r^2] \, f_{\alpha\beta}(r) \, dr$$

$$= \int_{-\infty}^{\infty} \hat{S}^\alpha \hat{S}^\beta \exp\left[-\sum_{\gamma=1}^{n} \left(\lambda - \frac{p(\Delta J)^2}{4} \lambda_\gamma \right) (\hat{S}^\gamma)^2 \right] d\hat{S}^1 \cdots d\hat{S}^n$$

$$= \frac{\pi^{n/2}}{2} \left(\lambda - \frac{p(\Delta J)^2}{4} \lambda_\alpha \right)^{-1} \prod_{\gamma=1}^{n} \left(\lambda - \frac{p(\Delta J)^2}{4} \lambda_\gamma \right)^{-1/2} \delta_{\alpha\beta} \tag{2.20}$$

where $\delta_{\alpha\beta}$ represents the Kronecker δ-symbol. In the same fashion we evaluate the corresponding Laplace transform for g and find

$$\int_0^\infty \exp[-\lambda x] \frac{g(\sqrt{x})}{2\sqrt{x}} dx = \pi^{n/2} \prod_{\gamma=1}^{n} \left(\lambda - \frac{p(\Delta J)^2}{4} \lambda_\gamma \right)^{-1/2} \tag{2.21}$$

By inverting the Laplace transforms given by Eqs. (2.20) and (2.21) and using our definitions of $f_{\alpha\beta}(\sqrt{n})$ and $g(\sqrt{n})$, we can then write the order parameter equation (2.17) in the following form:

$$O^T Q O = \left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp[\lambda n] \left(\lambda - \frac{p(\Delta J)^2}{4} \lambda_\alpha \right)^{-1} \right.$$
$$\times \left. \prod_{\gamma=1}^{n} \left(\lambda - \frac{p(\Delta) J^2}{4} \lambda_\gamma \right)^{-1/2} d\lambda \cdot \delta_{\alpha\beta} \right\}$$
$$\times \left[2 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp[\lambda n] \prod_{\gamma=1}^{n} \left(\lambda - \frac{p(\Delta J)^2}{4} \lambda_\gamma \right)^{-1/2} d\lambda \right]^{-1} \tag{2.22}$$

Since the right-hand side of this equation is a diagonal matrix, we find that the similarity transformation with O not only diagonalizes $Q^{(p-1)}$, as defined, but also Q. What is more, since our choice of O was arbitrary as long as Eq. (2.15) was satisfied, we actually find that every orthonormal similarity transformation which diagonalizes $Q^{(p-1)}$ must also diagonalize Q.

A theorem in matrix theory states that for arbitrary matrices A and B, if B commutes with every matrix which commutes with A, then B is a polynomial in A.[7] By a slight modification of the proof, one can show that if we have two symmetric matrices A and B, and if every orthonormal similarity transformation which diagonalizes A also diagonalizes B, then B is a polynomial in A. This means that Q must be a polynomial in $Q^{(p-1)}$. Furthermore, since $Q^{(p-1)}$ is symmetric, it is a simple matrix, and therefore the degree of its minimum polynomial is equal to the number r of distinct eigenvalues of $Q^{(p-1)}$. Thus, Q must be a polynomial in $Q^{(p-1)}$ of maximum degree $r-1$.

If we now denote by $\mu_1,..., \mu_n$ the eigenvalues of Q and as before by $\lambda_1,..., \lambda_n$ the eigenvalues of $Q^{(p-1)}$, then Eq. (2.22) and the above remarks lead us to the final form of the order parameter equations:

$$\mu_\alpha = \left\{ \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp[\lambda n] \left(\lambda - \frac{p(\Delta J)^2}{4} \lambda_\alpha \right)^{-1} \prod_{\gamma=1}^{n} \left(\lambda - \frac{p(\Delta J)^2}{4} \lambda_\gamma \right)^{-1/2} d\lambda \right\}$$
$$\times \left\{ 2 \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp[\lambda n] \prod_{\gamma=1}^{n} \left(\lambda - \frac{p(\Delta J)^2}{4} \lambda_\gamma \right)^{-1/2} d\lambda \right\}^{-1}, \qquad \alpha = 1,..., n \tag{2.23}$$

and

$$Q = a_0 I + a_1 Q^{(p-1)} + \cdots + a_{r-1} [Q^{(p-1)}]^{r-1} \tag{2.24}$$

where r is the number of distinct eigenvalues of $Q^{(p-1)}$, I represents the unit matrix, and the a_i are some real numbers. We further have the constraint

$$q_{\alpha\beta} \geqslant 0 \qquad \text{if} \quad p = \text{odd} \tag{2.25}$$

imposed by our model on the order parameters $q_{\alpha\beta}$.

The challenge now is to find a matrix solution Q of Eqs. (2.12) and (2.13) or equivalently of Eqs. (2.23)–(2.25).

The case $n = 1$ is trivial. Equation (2.5) combined with the normalization condition $\|\mathbf{S}\| = \sqrt{n}$ already dictates that the (one-dimensional) matrix Q simply equals 1 in this case. By inserting this into Eq. (2.7) and finally into Eq. (2.8), we get

$$\langle Z_N \rangle \xrightarrow{N \to \infty} \exp\left[\left(\frac{(\Delta J)^2}{4} + \ln 2 \right) N \right] \tag{2.26}$$

This result, which was obtained by the saddle point method above, is in perfect agreement with the result we get by evaluating the trace in Eq. (2.4) directly (which is possible for $n = 1$).

All cases $n > 1$ are nontrivial. We are going to present the complete solution for the $n = 2$ case in the next section and for the $n = 3$ case in Section 4. For general n, we shall derive explicit forms of the order parameter equations for all stable solutions in Section 5.

One final remark about the case $p = 2$. The case $p = 2$ is special since the order parameter equation (2.24) can always be satisfied by choosing $a_1 \equiv 1$, and $a_i \equiv 0$ if $i \neq 1$. Furthermore, we have $\mu_\gamma = \lambda_\gamma$, which means that we only have to find a set of eigenvalues μ_γ satisfying Eq. (2.23). All order parameter matrices Q that have this same set of eigenvalues, i.e., all matrices which are generated from a diagonal matrix consisting of these eigenvalues by an orthonormal similarity transformation, will then be a legitimate solution.

The only task for $p=2$, therefore, consists of finding a diagonal matrix

$$Q_d = \mathrm{diag}(\mu_1, \dots, \mu_n) \tag{2.27}$$

which solves the order parameter equation (2.23). The most general solution of the order parameter equations (2.23) and (2.24) is then given by

$$Q = O^T Q_d O \tag{2.28}$$

where O is an arbitrary orthonormal matrix. This corresponds to the invariance of the annealed partition function (2.3) under orthonormal transformations of the spin vectors $\mathbf{S}_i$ when $p=2$.

3. $n=2$ MODEL

3.1. Solutions

The order parameter matrix Q in this case is two dimensional, consisting of the order parameters q_{11}, q_{12}, q_{21}, and q_{22}.

However, these order parameters are not independent. Our normalization condition for n-vectors requires that

$$(S^1)^2 + (S^2)^2 = 2 \tag{3.1}$$

This equation imposes constraints on the vector components S^1 and S^2,

$$0 \leqslant (S^\alpha)^2 \leqslant 2 \tag{3.2}$$

$$-1 \leqslant S^1 S^2 \leqslant 1 \tag{3.3}$$

Because of Eq. (2.5), these constraints on the vector components then translate into the following constraints for the order parameters:

$$q_{22} = 2 - q_{11} \tag{3.4}$$

$$q_{21} = q_{12} \tag{3.5}$$

$$0 \leqslant q_{11} \leqslant 2 \tag{3.6}$$

$$-1 \leqslant q_{12} \leqslant 1 \quad \text{if} \quad p = \text{even} \tag{3.7}$$

$$0 \leqslant q_{12} \leqslant 1 \quad \text{if} \quad p = \text{odd} \tag{3.8}$$

where we have incorporated the constraint imposed by Eq. (2.25) into the last equation.

The most general order parameter matrix Q which solves the order parameter equations (2.23)–(2.25) must therefore be of the form

$$Q = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & (2-q_{11}) \end{pmatrix} \tag{3.9}$$

and has the eigenvalues

$$\mu_{1,2} = 1 \pm [(q_{11}-1)^2 + q_{12}^2]^{1/2} \tag{3.10}$$

The corresponding matrix $Q^{(p-1)}$ has the eigenvalues

$$\lambda_{1,2} = \tfrac{1}{2}(q_{11}^{p-1} + (2-q_{11})^{p-1} \pm \{[q_{11}^{p-1} - (2-q_{11})^{p-1}]^2 + 4q_{12}^{2(p-1)}\}^{1/2}) \tag{3.11}$$

We are now going to determine the order parameter equations.

The right hand sides of the order parameter equations (2.23) for $n=2$ are evaluated in Appendix A. By inserting the expressions (A10), (A12), and (A14) into Eq. (2.23) we get

$$\mu_1 = 1 - \frac{I_1(\frac{1}{4}p(\Delta J)^2(\lambda_2-\lambda_1))}{I_0(\frac{1}{4}p(\Delta J)^2(\lambda_2-\lambda_1))} \tag{3.12}$$

$$\mu_2 = 1 + \frac{I_1(\frac{1}{4}p(\Delta J)^2(\lambda_2-\lambda_1))}{I_0(\frac{1}{4}p(\Delta J)^2(\lambda_2-\lambda_1))} \tag{3.13}$$

We see that $\mu_1 + \mu_2 = 2$, as expected from Eq. (3.4) and the invariance of the trace of a matrix under orthonormal similarity transformations. This means that Eqs. (3.12) and (3.13) are not independent. If we can satisfy Eq. (3.12) for some q_{11} and q_{12}, then Eq. (3.13) will be satisfied automatically.

The order parameter equation (2.24) on the other hand tells us that for $n=2$ we have

$$Q = a_0 I \qquad (\lambda_1 = \lambda_2) \tag{3.14}$$

$$Q = a_0 I + a_1 Q^{(p-1)} \qquad (\lambda_1 \neq \lambda_2) \tag{3.15}$$

Inserting the expressions (3.10) and (3.11) for the eigenvalues into Eq. (3.12) gives

$$[(q_{11}-1)^2 + q_{12}^2]^{1/2} = \frac{I_1\{\frac{1}{4}p(\Delta J)^2[(q_{11}^{p-1} - (2-q_{11})^{p-1})^2 + 4q_{12}^{2(p-1)}]^{1/2}\}}{I_0\{\frac{1}{4}p(\Delta J)^2[(q_{11}^{p-1} - (2-q_{11})^{p-1})^2 + 4q_{12}^{2(p-1)}]^{1/2}\}} \tag{3.16}$$

and writing out the components of Eqs. (3.14) and (3.15) gives, for $\lambda_1 = \lambda_2$

$$\begin{aligned} q_{11} &= a_0 \\ 2 - q_{11} &= a_0 \\ q_{12} &= 0 \end{aligned} \tag{3.17}$$

and for $\lambda_1 \neq \lambda_2$

$$q_{11} = a_0 + a_1 q_{11}^{p-1} \tag{3.18}$$

$$2 - q_{11} = a_0 + a_1(2 - q_{11})^{p-1} \tag{3.19}$$

$$q_{12} = a_1 q_{12}^{p-1} \tag{3.20}$$

Finally, we have the constraint imposed by the order parameter equation (2.25)

$$q_{12} \geqslant 0 \qquad \text{if} \quad p = \text{odd} \tag{3.21}$$

Equations (3.16)–(3.21) constitute the order parameter equations for the case $n = 2$. We are now going to solve them.

3.1.1. Case of One Distinct Eigenvalue λ. In this case $\lambda_1 = \lambda_2$ and Eq. (3.17) already dictates

$$q_{11} = q_{22} = 1, \qquad q_{12} = 0 \tag{3.22}$$

It is easy to see that this solution is also consistent with Eq. (3.16) for all ΔJ, i.e., all temperatures T, and all p since $I_1(0) = 0$ and $I_0(0) = 1$. Our stability analysis, however, will reveal that $q_{11} = q_{22} = 1$, $q_{12} = 0$ represents only the high-temperature solution.

3.1.2. Case of Two Distinct Eigenvalues λ_1 and λ_2. In this case we have $\lambda_1 \neq \lambda_2$ and we have to distinguish three different cases.

Case a. $q_{11} \neq 1$ and $q_{12} = 0$. Equation (3.20) is automatically satisfied when $q_{12} = 0$. Since $q_{11} \neq 1$, we can further satisfy Eqs. (3.18) and (3.19) by choosing

$$a_0 \equiv q_{11} + \frac{2(q_{11} - 1) q_{11}^{p-1}}{q_{11}^{p-1} - (2 - q_{11})^{p-1}} \tag{3.23}$$

$$a_1 \equiv \frac{2(q_{11} - 1)}{q_{11}^{p-1} - (2 - q_{11})^{p-1}} \tag{3.24}$$

Thus, the only equation which remains to be satisfied is Eq. (3.16). It now becomes

$$q_{11} - 1 = \frac{I_1[\frac{1}{4}p(\Delta J)^2 (q_{11}^{p-1} - (2-q_{11})^{p-1})]}{I_0[\frac{1}{4}p(\Delta J)^2 (q_{11}^{p-1} - (2-q_{11})^{p-1})]} \tag{3.25}$$

where we were able to omit any moduli since $q_{11} = (1 + (q_{11} - 1))$, $(2 - q_{11}) = (1 - (q_{11} - 1))$, and since $I_1(-z) = -I_1(z)$, $I_0(-z) = I_0(z)$. This is a transcendental equation which can be solved numerically. We see immediately that whenever $q_{11} = 1 + \Delta q$ is a solution, then $q_{11} = 1 - \Delta q$ will be a solution as well. This simply means that we can interchange the roles of q_{11} and q_{22}, which in turn is simply a manifestation of the symmetry of the annealed partition function (2.3) under interchange of coordinates S_i^1 and S_i^2.

A numerical study of Eq. (3.25) shows that no solutions exist for small ΔJ, i.e., at high temperatures T.

For $p = 2$, 3, 4 we find one ΔJ_c such that for all $\Delta J > \Delta J_c$, i.e., for all low temperatures $T < T_c$, we have exactly one solution $1 < q_{11} < 2$ and one solution $0 < q_{11} < 1$ corresponding to the above-mentioned interchangeability of q_{11} and q_{22}.

For all $p > 4$ there are two transition temperatures. For $\Delta J_{c_1} < \Delta J < \Delta J_{c_2}$, i.e., for low temperatures $T_{c_1} > T > T_{c_2}$, we have two solutions $1 < q_{11_b} < q_{11_a} < 2$ and two solutions $0 < q_{11_a} < q_{11_b} < 1$ (corresponding to the interchangeability of q_{11} and q_{22}). When $\Delta J > \Delta J_{c_2}$, i.e., $T < T_{c_2}$, there exists only one solution $1 < q_{11} < 2$ and one solution $0 < q_{11} < 1$.

The phase transition points ΔJ_c and ΔJ_{c_2} can be determined analytically as follows. The right-hand side (rhs) of Eq. (3.25) is equal to 0 for $q_{11} = 1$. For $q_{11} > 1$, it is always positive and bounded from above since $I_1/I_0 < 1$. The left-hand side (lhs) of Eq. (3.25) constitutes a straight line of slope 1 through $q_{11} = 1$. Thus, we shall always get at least one intersection of the lhs of Eq. (3.25) with the rhs if the derivative of the rhs at $q_{11} = 1$ is greater than 1

$$\frac{\partial}{\partial q_{11}}(\text{rhs})\bigg|_{q_{11}=1} = \frac{p(p-1)(\Delta J)^2}{4} > 1 \tag{3.26}$$

This relation will be satisfied for all ΔJ greater than the critical value

$$\Delta J_c = \left[\frac{4}{p(p-1)}\right]^{1/2} \tag{3.27}$$

It turns out that this critical value represents exactly the ΔJ_c we found above for $p = 2$, 3, 4, and it represents ΔJ_{c_2} for $p > 4$

$$\Delta J_{c_2} = \left[\frac{4}{p(p-1)}\right]^{1/2} \tag{3.28}$$

Although there is not a simple analytical expression for ΔJ_{c_1}, we always have

$$\Delta J_{c_1} < \left[\frac{4}{p(p-1)}\right]^{1/2} \tag{3.29}$$

Our stability analysis in Section 3.2 will show that the solutions q_{11}, q_{22} and q_{11_a}, q_{22_a} represent the low-temperature states of our model, whereas the solutions q_{11_b} and q_{22_b} are unstable.

A final note regarding the case $p=2$. We have just determined the diagonal solution Q_d of the order parameter equation (2.23) for $n=2$ and $p=2$. As we have shown in Eqs. (2.27) and (2.28), the most general solution for the order parameter equations (3.16)–(3.21) in this case is then given by

$$Q = O^T \cdot \operatorname{diag}(\mu_1, 2-\mu_1) \cdot O \tag{3.30}$$

where O is an arbitrary orthonormal 2×2 matrix and where μ_1 coincides with the order parameter q_{11} of the matrix Q_d which we obtained above. Since the most general orthonormal 2×2 matrix O is of the form

$$O \equiv \begin{pmatrix} \cos(\phi/2) & -\sin(\phi/2) \\ \sin(\phi/2) & \cos(\phi/2) \end{pmatrix} \tag{3.31}$$

we therefore find that the most general solution of the order parameter equations (3.16)–(3.21) when $p=2$ is given by

$$\begin{aligned} q_{11} &= 1 + (\mu_1 - 1)\cos(\phi) \\ q_{22} &= 1 - (\mu_1 - 1)\cos(\phi) \\ q_{12} &= (\mu_1 - 1)\sin(\phi) \end{aligned} \tag{3.32}$$

where ϕ is an arbitrary angle. The stability analysis in Section 3.2 will show that this solution is stable.

Case b. $q_{11}=1$, $q_{12}\neq 0$, and $p>2$. Since $q_{12}\neq 0$ and $q_{11}=1$, we can satisfy the order parameter equations (3.18)–(3.20) simultaneously by choosing

$$a_0 \equiv 1 - q_{12}^{2-p} \tag{3.33}$$

$$a_1 \equiv q_{12}^{2-p} \tag{3.34}$$

The only equation which then remains to be satisfied is Eq. (3.16). It now becomes

$$|q_{12}| = \frac{I_1(\frac{1}{4}p(\Delta J)^2\,|q_{12}|^{p-1})}{I_0(\frac{1}{4}p(\Delta J)^2\,|q_{12}|^{p-1})} \tag{3.35}$$

Again this is a transcendental equation which can be solved numerically.

A numerical study shows that no solutions exist for small ΔJ, i.e., at high temperatures T.

For all $p>2$ we find one $\Delta J_{c_3} > \Delta J_c$ or, respectively, one $\Delta J_{c_3} > \Delta J_{c_2} > \Delta J_{c_1}$, such that for all $\Delta J > \Delta J_{c_3}$, i.e., all very low temperatures $T < T_{c_3}$, we have two solutions $0 < q_{12_b} < q_{12_a} < 1$ and two solutions $-1 < q_{12_a} < q_{12_b} < 0$. The negative solutions are simply obtained by changing the sign of the positive solutions and arise from the symmetry of Eq. (3.35). Negative solutions for $p = \text{odd}$ do not exist because of Eq. (2.25). Although there is no simple analytical expression for ΔJ_{c_3}, we always have

$$\Delta J_{c_3} > \left[\frac{4}{p(p-1)}\right]^{1/2} \tag{3.36}$$

Our stability analysis in Section 3.2 will show that all solutions q_{12_a} and q_{12_b} are unstable.

Case c. $q_{11} \neq 1$, $q_{12} \neq 0$, and $p>2$. Since $q_{11} \neq 1$, Eqs. (3.18) and (3.19) have the unique solutions

$$a_0 \equiv q_{11} + \frac{2(q_{11}-1)\, q_{11}^{p-1}}{q_{11}^{p-1} - (2-q_{11})^{p-1}} \tag{3.37}$$

$$a_1 \equiv \frac{2(q_{11}-1)}{q_{11}^{p-1} - (2-q_{11})^{p-1}} \tag{3.38}$$

By inserting Eq. (3.38) into the order parameter equation (3.20), and since we require $q_{12} \neq 0$, we get

$$\begin{aligned} q_{12} &= \left[\frac{q_{11}^{p-1} - (2-q_{11})^{p-1}}{2(q_{11}-1)}\right]^{1/(p-2)} \\ &= \left[\frac{[1+(q_{11}-1)]^{p-1} - [1-(q_{11}-1)]^{p-1}}{2(q_{11}-1)}\right]^{1/(p-2)} \\ &= \left[(p-1) + \sum_{k>1,\,\text{odd}}^{p-1} \binom{p-1}{k} (q_{11}-1)^{k-1}\right]^{1/(p-2)} \geqslant (p-1)^{1/(p-2)} \end{aligned} \tag{3.39}$$

For $p>2$, this is in contradiction to the constraint $q_{12} \leqslant 1$, Eqs. (3.7) and (3.8), imposed on q_{12} by our model. Thus, there are no solutions $q_{11} \neq 1$, $q_{12} \neq 0$ when $p>2$.

3.2. Stationarity of the Free Energy

The Lagrange multipliers $\lambda_{\alpha\beta}$ in Eqs. (2.7) and (2.10) are only auxiliary parameters which help to evaluate the free energy at the equilibrium configuration of the parameters $q_{\alpha\beta}$. If we knew how to evaluate the free energy without the detour of the auxiliary $\lambda_{\alpha\beta}$, we should obtain exactly the same free energy as a function of the same equilibrium configuration of the parameters $q_{\alpha\beta}$, but without any $\lambda_{\alpha\beta}$. Thus, the only physical parameters of the free energy are the $q_{\alpha\beta}$.

Stationarity of the free energy for our annealed system now means that its free energy is a minimum with respect to fluctuations of the spin configuration about the equilibrium point, i.e., with respect to fluctuations of the order parameters $q_{\alpha\beta}$ about their equilibrium values. However, from Eqs. (3.4) and (3.5) we have the constraints $q_{22}=2-q_{11}$ and $q_{21}=q_{12}$ resulting from our normalization condition for n-vectors and from our definition (2.5) of $q_{\alpha\beta}$. The order parameters $q_{\alpha\beta}$ can therefore only fluctuate within these two constraints imposed by the model, and the free energy becomes a function of two independent order parameters q_{11} and q_{12}.

From Eqs. (2.7), (2.8), and (2.12) we then obtain the free energy per spin for the $n=2$ model as

$$
\begin{aligned}
\frac{a}{kT} &= \lim_{N\to\infty}\left(-\frac{1}{N}\ln\langle Z_N\rangle\right)\\
&= -G^*\\
&= \frac{(p-1)(\Delta J)^2}{4}\,[q_{11}^p+(2-q_{11})^p+2\,|q_{12}|^p]\\
&\quad -\ln\int_{\|S\|=\sqrt{2}}\exp\left\{\frac{p(\Delta J)^2}{4}\,[q_{11}^{p-1}(S^1)^2+(2-q_{11})^{p-1}\,(S^2)^2]\right.\\
&\quad \left.+\frac{p(\Delta J)^2}{2}\,|q_{12}|^{p-1}\,S^1S^2\right\}dS^1\,dS^2 \qquad (3.40)\\
&= \frac{(p-1)(\Delta J)^2}{4}\,[q_{11}^p+(2-q_{11})^p+2\,|q_{12}|^p]\\
&\quad -\frac{p(\Delta J)^2}{4}\,[q_{11}^{p-1}+(2-q_{11})^{p-1}]\\
&\quad -\ln I_0\left(\frac{p(\Delta J)^2}{4}\,\{[q_{11}^{p-1}-(2-q_{11})^{p-1}]^2+4q_{12}^{2(p-1)}\}^{1/2}\right)-\ln 2\pi\sqrt{2} \qquad (3.41)
\end{aligned}
$$

where we have made the coordinate transformation (2.15) and used Eqs. (2.18), (2.21), and (A10) in the evaluation of the integral in the last step.

The free energy (3.41) will be stationary (stable) if it constitutes a local minimum with respect to fluctuations δ_{11} and δ_{12} about the solution points q_{11} and q_{12}. This has to be investigated for all possible solutions of the order parameter equations which we obtained in the previous section.

Case a. $q_{11}=1$ and $q_{12}=0$. The free energy in Eq. (3.41) will be stable if its Hessian matrix with respect to q_{11} and q_{12} is positive definite. Evaluating the Hessian matrix

$$H \equiv \begin{pmatrix} \dfrac{\partial^2}{\partial q_{11}\,\partial q_{11}} \dfrac{a}{kT} & \dfrac{\partial^2}{\partial q_{11}\,\partial q_{12}} \dfrac{a}{kT} \\ \dfrac{\partial^2}{\partial q_{11}\,\partial q_{12}} \dfrac{a}{kT} & \dfrac{\partial^2}{\partial q_{12}\,\partial q_{12}} \dfrac{a}{kT} \end{pmatrix} \tag{3.42}$$

at the solution point $q_{11}=1$, $q_{12}=0$ yields the two eigenvalues

$$\xi_1 = \frac{p(p-1)(\Delta J)^2}{2}\left[1 - \frac{p(p-1)(\Delta J)^2}{4}\right] \tag{3.43}$$

$$\xi_2 = \begin{cases} \frac{3}{4}(\Delta J)^2\,[2-(\Delta J)^2], & p=2 \\ 0, & p>2 \end{cases} \tag{3.44}$$

This means that for $p=2$ the solution (3.22) is stable if $\Delta J<\sqrt{2}$ and unstable if $\Delta J>\sqrt{2}$.

For $p>2$ the solution (3.22) is unstable if $\Delta J>\{4/[p(p-1)]\}^{1/2}$. If $\Delta J<\{4/[p(p-1)]\}^{1/2}$, however, the Hessian becomes positive semidefinite and leaves us in aporia. We then have to look for higher-order fluctuations. One can show that for $p>2$ and infinitesimal fluctuations $\delta_{\alpha\beta}$ we have

$$\begin{aligned} &\ln \int_{\|\mathbf{S}\|=\sqrt{n}} \exp\left[\frac{p(\Delta J)^2}{4} \sum_{\alpha=1}^{n} q_{\alpha\alpha}^{p-1}(S^\alpha)^2 + \frac{p(\Delta J)^2}{2} \sum_{\alpha<\beta} \delta_{\alpha\beta}^{p-1} S^\alpha S^\beta\right] d\mathbf{S} \\ &\quad = \ln \int_{\|\mathbf{S}\|=\sqrt{n}} \exp\left[\frac{p(\Delta J)^2}{4} \sum_{\alpha=1}^{n} q_{\alpha\alpha}^{p-1}(S^\alpha)^2\right] d\mathbf{S} + \sum_{\alpha<\beta} O(\delta_{\alpha\beta}^{2p-2}) \end{aligned} \tag{3.45}$$

By using this relation, we find the following expansion for the free energy (3.40), (3.41) about the solution point $q_{11}=1$, $q_{12}=0$:

$$\begin{aligned} \frac{a}{kT} &= -\left[\frac{(\Delta J)^2}{2} + \ln 2\pi\sqrt{2}\right] + \frac{1}{2}\xi_1 \delta_{11}^2 \\ &\quad + \frac{(p-1)(\Delta J)^2}{2}\,|\delta_{12}|^p + O(\delta_{11}^4) + O(\delta_{12}^{2p-2}) \end{aligned} \tag{3.46}$$

where ξ_1 is given by Eq. (3.43). From this expansion we see that for all $p>2$ stability arises from pth-order fluctuations in q_{12} if $\Delta J<\{4/[p(p-1)]\}^{1/2}$. We shall call this tenuous stability "pth-order stability."

In summary, the solution $q_{11}=1$, $q_{12}=0$ is stable if $\Delta J<\{4/[p(p-1)]\}^{1/2}$, i.e., if $T>T_c$, and becomes unstable if $\Delta J>\{4/[p(p-1)]\}^{1/2}$, i.e., if $T<T_c$. Thus, it represents the high-temperature solution for all p.

Case b. $q_{11}\neq 1$ and $q_{12}=0$.

$p=2$: In this case the free energy from Eq. (3.40) becomes

$$\frac{a}{kT}=\frac{(\Delta J)^2}{4}\sum_{\alpha,\beta=1}^{2} q_{\alpha\beta}^2-\ln\int_{\|\mathbf{S}\|=\sqrt{2}}\exp\left[\frac{(\Delta J)^2}{2}\mathbf{S}^T Q\mathbf{S}\right]d\mathbf{S} \tag{3.47}$$

$\sum_{\alpha,\beta=1}^{2} q_{\alpha\beta}^2$ represents the square of the Euclidean matrix norm for the matrix Q. It is easy to show that the Euclidean matrix norm is invariant under orthonormal similarity transformations. Further, the integral on the right-hand side of Eq. (3.47) is invariant under orthonormal similarity transformations, as we saw in Section 2. Thus, for $p=2$, a/kT is invariant under orthonormal similarity transformations of the matrix Q. This corresponds to the invariance of the annealed partition function (2.3) under orthonormal transformations of the spin vectors $\mathbf{S}_i$ when $p=2$. Finally, we have the constraint $\mu_2=2-\mu_1$ resulting from our normalization condition for n-vectors.

The stability of a/kT is therefore completely determined by the fluctuations of the eigenvalue μ_1 about its equilibrium value. Rewriting Eq. (3.47) in terms of the eigenvalue μ_1 gives

$$\begin{aligned}\frac{a}{kT}&=\frac{(\Delta J)^2}{4}[\mu_1^2+(2-\mu_1)^2]\\&\quad-\ln\int_{\|\mathbf{S}\|=\sqrt{2}}\exp\left\{\frac{(\Delta J)^2}{2}[\mu_1(S^1)^2+(2-\mu_1)(S^2)^2]\right\}d\mathbf{S}\\&=-\ln 2\pi\sqrt{2}-\frac{(\Delta J)^2}{2}+\frac{(\Delta J)^2}{2}(\mu_1-1)^2-\ln I_0[(\Delta J)^2(\mu_1-1)]\end{aligned} \tag{3.48}$$

where we have used Eqs. (2.18), (2.21), and (A10) in the evaluation of the integral in the last step. This free energy will be a minimum with respect to fluctuations δ_1 about the equilibrium values μ_1 if its second derivative at μ_1 is positive. By taking the second derivative of Eq. (3.48), we get

$$\frac{\partial^2}{\partial\mu_1^2}\frac{a}{kT}=(\Delta J)^4(\mu_1-1)^2+(\Delta J)^2[2-(\Delta J)^2] \tag{3.49}$$

where we have used the fact that

$$\mu_1 - 1 = \frac{I_1[(\Delta J)^2 (\mu_1 - 1)]}{I_0[(\Delta J)^2 (\mu_2 - 1)]} \tag{3.50}$$

from Eq. (3.25) when $p = 2$.

A numerical plot of the right-hand side of Eq. (3.49) for the solutions μ_1 obtained in the previous section shows that $\partial^2/\partial\mu_1^2(a/kT) > 0$ for the entire low-temperature continuum $\Delta J > \sqrt{2}$. Combined with our remarks above, this means that the low-temperature solution (3.32) for $p = 2$ is stable for all $\Delta J > \sqrt{2}$, i.e., for all $T < T_c$.

$p > 2$: Evaluating the Hessian for the free energy (3.41) at the solution points q_{11} given by Eq. (3.25) and $q_{12} = 0$ shows that one eigenvalue is identically equal to 0, for all p's. Thus, the Hessian is semidefinite and cannot give us a conclusive answer with respect to the stability of our solutions. We therefore have to look again at higher-order fluctuations.

By using the relation (3.45), we find the following expansion for the free energy (3.40), (3.41) about the solution points $q_{11} \neq 0$, $q_{12} = 0$:

$$\begin{aligned} \frac{a}{kT} = & -\ln 2\pi\sqrt{2} - \frac{p(\Delta J)^2}{4}\,[q_{11}^{p-1} + (2 - q_{11})^{p-1}] \\ & + \frac{(p-1)(\Delta J)^2}{4}\,[q_{11}^p + (2 - q_{11})^p] \\ & - \ln I_0\left\{\frac{p(\Delta J)^2}{4}\,[q_{11}^{p-1} - (2 - q_{11})^{p-1}]\right\} \\ & + \frac{1}{2}\xi_1\delta_{11}^2 + \frac{(p-1)(\Delta J)^2}{4}\,|\delta_{12}|^p + O(\delta_{11}^4) + O(\delta_{12}^{2p-2}) \end{aligned} \tag{3.51}$$

q_{11} is hereby a solution of Eq. (3.25). ξ_1 is the first eigenvalue of the Hessian (3.42) and is now given by

$$\begin{aligned} \xi_1 = & \frac{p(p-1)(\Delta J)^2}{4}\,[q_{11}^{p-2} + (2 - q_{11})^{p-2}]\left\{1 - \frac{p(p-1)(\Delta J)^2}{4}\right. \\ & \times [q_{11}^{p-2} + (2 - q_{11})^{p-2}] \\ & \left.\times\left[1 - (q_{11} - 1)\left\{\frac{p(\Delta J)^2}{4}\,[q_{11}^{p-1} - (2 - q_{11})^{p-1}]\right\}^{-1} - (q_{11} - 1)^2\right]\right\} \end{aligned} \tag{3.52}$$

A numerical plot of ξ_1 for the solutions q_{11} obtained in the previous section shows that ξ_1 is positive for all q_{11} when $p = 3$, 4 and for the solutions q_{11_a} when $p>4$. For the solutions q_{11_b}, on the other hand, ξ_1 is negative. This means that for $p=3$, 4, all low-temperature solutions q_{11} are stable. For $p>4$, only the low-temperature solutions q_{11_a} are stable. Again, the stability arises from pth-order fluctuations in q_{12}.

Case c. $q_{11}=1$, $q_{12}\neq 0$, and $p>2$. Evaluating the Hessian matrix (3.42) for the free energy (3.41) at the solution points $q_{11}=1$ and q_{12} given by Eq. (3.35) yields the two eigenvalues

$$\xi_1=\frac{p(p-1)(\Delta J)^2}{2}\left(1-\frac{p-1}{|q_{12}|^{p-2}}\right) \tag{3.53}$$

$$\xi_2=\frac{p^2(p-1)(\Delta J)^2}{2}|q_{12}|^{p-2}\left[1-\frac{(p-1)(\Delta J)^2}{2}|q_{12}|^{p-2}(1-q_{12}^2)\right] \tag{3.54}$$

where we have used Eq. (3.35) in the evaluation of the right-hand sides.

In the previous section we have seen that $|q_{12}|\leqslant 1$ for all p; see Eqs. (3.7) and (3.8). Thus, for $p>2$ the eigenvalue ξ_1 will be negative. There are no stable solutions with $q_{12}\neq 0$ when $p>2$.

3.3. Summary

The free energy per spin for the $n=2$ model was derived in Eq. (3.41) in the thermodynamic limit $N\to\infty$ by the method of steepest descent from Eq. (2.6). It will only hold for stable solutions of the order parameter equations. If we find more than one stable solution at a certain temperature T (as is the case for $p>4$), the solution which yields the lowest free energy (3.41) will constitute the true equilibrium configuration. This makes both physical sense and arises mathematically from choosing the dominant saddle point in the evaluation of Eq. (2.6) by the method of steepest descent.

For $p>4$ we found that between the temperatures T_{c_1} and T_{c_2} both the high-temperature solution $q_{11}=q_{22}=1$, $q_{12}=0$ and the "antisymmetric" diagonal solution $q_{11_a}\neq q_{22_a}$, $q_{12}=0$ are stable. A numerical comparison of the free energy (3.41) for these two solutions for various $p>4$ shows that in all cases we find exactly one ΔJ_c with $\Delta J_{c_1}<\Delta J_c<\Delta J_{c_2}$. For $\Delta J<\Delta J_c$, i.e., for $T>T_c$, the high-temperature free energy remains the lower free energy, whereas for $\Delta J>\Delta J_c$, i.e., for $T<T_c$, the "antisymmetric" free energy has a lower value. Hence, even for $p>4$, we actually have only one phase transition at a certain $\Delta J_c<\{4/[p(p-1)]\}^{1/2}$.

Combined with the results from the previous sections, this allows us to describe the various cases for the $n=2$ model as follows.

3.3.1. Case $p=2$. The model can be described by one order parameter μ_1 and we have one phase transition at $\Delta J_c = \{4/[p(p-1)]\}^{1/2} = \sqrt{2}$.

For $\Delta J > \Delta J_c$, i.e., $T < T_c$, the order parameter $1 < \mu_1 < 2$ is a solution of Eq. (3.25). For $\Delta J < \Delta J_c$, i.e., $T > T_c$, μ_1 is identically equal to 1.

The two-dimensional order parameter matrix Q is given in terms of μ_1 by Eq. (3.32). The free energy per spin is given by Eq. (3.41).

Thus, for $T > T_c$ we have the high-temperature solution $q_{11} = q_{22} = 1$, $q_{12} = 0$.

At $T = T_c$ we have a phase transition. It is a first-order phase transition since $-\partial a/\partial T$ has a discontinuity at $T = T_c$. However, the spin configuration (q_{11}, q_{22}, q_{12}) is a continuous function of T at T_c.

For $T < T_c$, we obtain a degenerate low-temperature continuum of states which lie on a circle of radius $\mu_1 - 1$ about $q_{11} = 1$, $q_{12} = 0$ in configurational order parameter space. The degeneracy results from the invariance of the annealed partition function (2.3) under orthonormal transformations (two-dimensional rotations) of the spin vectors $\mathbf{S}_i$ when $p = 2$.

3.3.2. Case $p=3, 4$. The model can be described by one order parameter q_{11}, and we have one phase transition at $\Delta J_c = \{4/[p(p-1)]\}^{1/2}$.

For $\Delta J > \Delta J_c$, i.e., $T < T_c$, the order parameter $1 < q_{11} < 2$ is a solution of Eq. (3.25). For $\Delta J < \Delta J_c$, i.e., $T > T_c$, q_{11} is identically equal to 1.

The two-dimensional order parameter matrix Q is given by q_{11}, $q_{22} = 2 - q_{11}$, and $q_{12} = 0$. The free energy per spin is given by Eq. (3.41).

Thus, as for $p = 2$, we have the high-temperature solution $q_{11} = q_{22} = 1$, $q_{12} = 0$ when $T > T_c$.

At $T = T_c$ we have a phase transition. As for $p = 2$, it is a first-order phase transition with the spin configuration (q_{11}, q_{22}, q_{12}) being a continuous function of T at T_c.

For $T < T_c$, we obtain a diagonal "antisymmetric" solution $q_{11} \neq q_{22}$, $q_{12} = 0$ with twofold degeneracy. The degeneracy arises from the interchangeability of q_{11} and q_{22} which corresponds to the symmetry in the first and second vector components in the annealed partition function (2.3).

3.3.3. Case $p>4$. These models are almost completely analogous to the cases $p = 3$, 4. The (main) difference is that the phase transition at T_c is not only a first-order transition, but the spin configuration (q_{11}, q_{22}, q_{12}) displays a jump discontinuity at T_c as well: $q_{11} = 1$ for $T > T_c$, and $1 + \Delta < q_{11} < 2$ for some Δ when $T < T_c$.

Further, we do not have an analytic expression for ΔJ_c. However, we know that $\Delta J_c < \{4/[p(p-1)]\}^{1/2}$. Numerically we find $\Delta J_c = 0.43422$ for $p=5$ and $\Delta J_c = 0.31918$ for $p=6$.

Since $\Delta J_c < \{4/[p(p-1)]\}^{1/2}$, we find $\Delta J_c \to 0$, i.e., $T_c \to \infty$ as $p \to \infty$. From Eq. (3.25) we further see that for finite ΔJ and $p \to \infty$ the only possible configurations are $q_{11}=2$, $q_{22}=0$, $q_{12}=0$ and $q_{11}=0$, $q_{22}=2$, $q_{12}=0$. This is also what we expect physically as the ordering element of interactions becomes dominant when $p \to \infty$.

4. $n=3$ MODEL

4.1. Solutions

Because of Eq. (2.5), the three-dimensional order parameter matrix Q is symmetric. However, as in the case $n=2$, its order parameters q_{11}, q_{22}, q_{33}, q_{12}, q_{13}, and q_{23} are not independent. Our normalization condition for n-vectors, $\|\mathbf{S}_i\| = \sqrt{3}$, imposes constraints on the vector components S^1, S^2, and S^3, which, because of Eq. (2.5), then translate into the following constraints for the order parameters:

$$q_{33} = 3 - q_{11} - q_{22} \tag{4.1}$$

$$0 \leqslant q_{\alpha\alpha} \leqslant 3 \tag{4.2}$$

$$-1.5 \leqslant q_{\alpha\beta} \leqslant 1.5 \qquad \text{if} \quad p = \text{even} \tag{4.3}$$

$$0 \leqslant q_{\alpha\beta} \leqslant 1.5 \qquad \text{if} \quad p = \text{odd} \tag{4.4}$$

where we have incorporated the constraint imposed by Eq. (2.25) into the last equation. In accordance with our previous notation, Q has the eigenvalues μ_1, μ_2, and μ_3, while the corresponding matrix $Q^{(p-1)}$ has the eigenvalues λ_1, λ_2, and λ_3.

The right-hand sides of the order parameter equations (2.23) for $n=3$ are evaluated in Appendix A. By inserting the expressions (A17) and (A21) into Eq. (2.23), we get

$$\mu_\alpha = 3\,\frac{\int_0^1 \exp[-a_\alpha u](1-u)^{1/2} I_0(b_\alpha u)\,du}{\int_0^1 \exp[-a_\alpha u](1-u)^{-1/2} I_0(b_\alpha u)\,du}, \qquad \alpha = 1, 2, 3 \tag{4.5}$$

with

$$a_\alpha \equiv \frac{3p(\Delta J)^2}{8}\,(2\lambda_\alpha - \lambda_\beta - \lambda_\gamma) \tag{4.6}$$

$$b_\alpha \equiv \frac{3p(\Delta J)^2}{8}\,(\lambda_\beta - \lambda_\gamma) \tag{4.7}$$

One can easily show that $\mu_1+\mu_2+\mu_3=3$, as expected from Eq. (4.1) and the invariance of the trace of a matrix under orthonormal similarity transformations. This means that Eqs. (4.5) for $\alpha=1$, 2, 3 are not independent. If we can satisfy Eqs. (4.5) for $\alpha=1$ and $\alpha=2$, then the equation for $\alpha=3$ will be satisfied automatically.

The order parameter equation (2.24) for $n=3$, on the other hand, becomes

$$Q=a_0 I \qquad (\lambda_1=\lambda_2=\lambda_3) \tag{4.8}$$

$$Q=a_0 I+a_1 Q^{(p-1)} \qquad (\lambda_\alpha=\lambda_\beta\neq\lambda_\gamma) \tag{4.9}$$

$$Q=a_0 I+a_1 Q^{(p-1)}+a_2[Q^{(p-1)}]^2 \qquad (\lambda_1\neq\lambda_2\neq\lambda_3) \tag{4.10}$$

Finally, we have the constraint imposed by the order parameter equation (2.25)

$$q_{\alpha\beta}\geqslant 0 \qquad \text{if} \quad p=\text{odd} \tag{4.11}$$

Equations (4.5)–(4.11) constitute the order parameter equations for the case $n=3$. We are now going to solve them.

4.1.1. Case of One Distinct Eigenvalue λ. In this case we have $\lambda_1=\lambda_2=\lambda_3$ and Eq. (4.8) already dictates

$$q_{11}=q_{22}=q_{33}=1, \qquad q_{12}=q_{13}=q_{23}=0 \tag{4.12}$$

This solution is also consistent with Eq. (4.5) for all ΔJ, i.e., all temperatures T, and all p since $I_0(0)=1$ and $\mu_1=\mu_2=\mu_3=1$. Our stability analysis, however, will reveal that, as for $n=2$, the solution $q_{\alpha\alpha}=1$, $q_{\alpha\beta}=0$ represents only the high-temperature solution.

4.1.2. Case of Two Distinct Eigenvalues λ_α and λ_γ. In this case we have $\lambda_\alpha=\lambda_\beta\neq\lambda_\gamma$. Hence, $a_\alpha=b_\alpha$ and the integrals in the order parameter equation (4.5) can be further evaluated. This is done in Appendix A. By inserting the expressions (A22), (A26), and (A27) into Eq. (2.23), we get

$$\mu_\alpha=\mu_\beta=\frac{3}{2z}\left[\frac{1+2z}{2}-\left(\frac{z}{\pi}\right)^{1/2}\frac{e^z}{\operatorname{erfi}(\sqrt{z})}\right] \tag{4.13}$$

with

$$z\equiv\frac{3p(\Delta J)^2}{4}(\lambda_\gamma-\lambda_\alpha) \tag{4.14}$$

The corresponding order parameter equation for μ_γ yields nothing new. Equation (4.13) replaces the order parameter equations (4.5) when we have only two distinct eigenvalues.

Choosing $a_1 = 0$ in the order parameter equation (4.9) simply recovers the case of one distinct eigenvalue λ. Thus, we require

$$a_1 \neq 0 \tag{4.15}$$

For $p = 2$ we can always satisfy Eq. (4.9) by choosing $a_1 = 1$ and $a_0 = 0$. As we have shown in Section 2, Eqs. (2.27) and (2.28), when $p = 2$ it suffices to find a diagonal solution Q_d of Eq. (4.13). The most general solution Q is then obtained by an arbitrary orthonormal similarity transformation of Q_d.

For $p > 2$, the off-diagonal elements of the order parameter equation (4.9) are

$$q_{\alpha\beta} = a_1 q_{\alpha\beta}^{p-1} \tag{4.16}$$

Since $p > 2$ and $a_1 \neq 0$, $q_{\alpha\beta}$ can then only assume the values

$$q_{\alpha\beta} = q_o, 0 \tag{4.17}$$

for some q_o. Here we have also used Eq. (4.11).

The diagonal elements of the order parameter equation (4.9), on the other hand, are

$$f(q_{\alpha\alpha}) \equiv a_1 q_{\alpha\alpha}^{p-1} - q_{\alpha\alpha} + a_0 = 0 \tag{4.18}$$

The derivative $f'(q_{\alpha\alpha})$ of f can have at most one zero for $q_{\alpha\alpha} \geqslant 0$ and real. This means that Eq. (4.18) will have at most two real solutions $q_{\alpha\alpha} \geqslant 0$. The condition $q_{\alpha\alpha} \geqslant 0$ is required by Eq. (4.2).

The most general order parameter matrix Q which we have to investigate for $p > 2$ can therefore have at most two distinct diagonal elements and its off-diagonal elements can only take on the values 0 or q_o. Modulo constant orthonormal similarity transformations, which simply rearrange the diagonal elements of Q, this reduces the problem of finding Q to eight different cases:

Case 1: $q_{11} = q_{22}$, $q_{33} = 3 - 2q_{11}$, $q_{12} = q_{13} = q_{23} = 0$.

Case 2: $q_{11} = q_{22}$, $q_{33} = 3 - 2q_{11}$, $q_{12} = q_o$, $q_{13} = q_{23} = 0$, and $p > 2$.

Case 3: $q_{11} = q_{22}$, $q_{33} = 3 - 2q_{11}$, $q_{12} = q_{23} = 0$, $q_{13} = q_o$, and $p > 2$.

Case 4: $q_{11} = q_{22}$, $q_{33} = 3 - 2q_{11}$, $q_{12} = q_{13} = 0$, $q_{23} = q_o$, and $p > 2$.

Case 5: $q_{11} = q_{22}$, $q_{33} = 3 - 2q_{11}$, $q_{12} = q_{13} = q_o$, $q_{23} = 0$, and $p > 2$.

Case 6: $q_{11} = q_{22}$, $q_{33} = 3 - 2q_{11}$, $q_{12} = q_{23} = q_o$, $q_{12} = 0$, and $p > 2$.

Case 7: $q_{11} = q_{22}$, $q_{33} = 3 - 2q_{11}$, $q_{12} = 0$, $q_{13} = q_{23} = q_o$, and $p > 2$.

Case 8: $q_{11} = q_{22}$, $q_{33} = 3 - 2q_{11}$, $q_{12} = q_{13} = q_{23} = q_o$, and $p > 2$.

One can analyze these cases by using the following facts. Q and $Q^{(p-1)}$ have the same number of distinct eigenvalues, since Q is a polynomial in $Q^{(p-1)}$. Q is a linear polynomial in $Q^{(p-1)}$. For $n \geqslant 2$, the only possible real solutions of

$$(a+b)^n = a^n + b^n \tag{4.19}$$

are

$$\begin{cases} a=0 \quad \text{or} \quad b=0 & (n=\text{even}) \\ a=0 \quad \text{or} \quad b=0 \quad \text{or} \quad a=-b & (n=\text{odd}) \end{cases} \tag{4.20}$$

For $n \geqslant 2$, the only possible real solutions of

$$(a+b)^n = a^n - b^n \tag{4.21}$$

are

$$\begin{cases} b=0 \quad \text{or} \quad a=-b & (n=\text{even}) \\ b=0 & (n=\text{odd}) \end{cases} \tag{4.22}$$

In order to get two identical eigenvalues μ, the characteristic polynomial of Q

$$\text{ch}(\mu) \equiv -\mu^3 + 3\mu^2 - b\mu + c \tag{4.23}$$

must satisfy the condition

$$c = \pm 2(1-b/3)^{3/2} + b - 2 \tag{4.24}$$

The detailed analysis of cases 1–8 is lengthy and therefore we shall merely state the results here.

There are only two possible solutions Q of the order parameter equations (4.9) and (4.13) for $T \geqslant 0$. In addition, there are four singular solutions which exist only at $T=0$.

The first nonsingular solution is the, for $p>2$ diagonal, solution

$$\begin{aligned} Q &= P^T \cdot \text{diag}(\mu_1, \mu_1, 3-2\mu_1) \cdot P \qquad (p>2) \\ Q &= O^T \cdot \text{diag}(\mu_1, \mu_1, 3-2\mu_1) \cdot O \qquad (p=2) \end{aligned} \tag{4.25}$$

where the similarity transformation with P represents an arbitrary permutation of the diagonal elements of diag($\cdots$) and where O is an arbitrary orthonormal matrix. The order parameter μ_1 is a solution of

$$\mu_1 = \frac{3}{2z}\left[\frac{1+2z}{2} - \left(\frac{z}{\pi}\right)^{1/2} \frac{e^z}{\text{erfi}(\sqrt{z})}\right] \tag{4.26}$$

with

$$z = \frac{3p(\Delta J)^2}{4} \left[(3-2\mu_1)^{p-1} - \mu_1^{p-1}\right] \tag{4.27}$$

This is a transcendental equation which can be solved numerically.

A numerical study of Eq. (4.26) shows that no solutions exist for small ΔJ, i.e., at high temperatures T.

For all p we find one ΔJ_{c_1} such that for all $\Delta J > \Delta J_{c_1}$, i.e., for all low temperatures $T < T_{c_1}$, we have exactly two solutions $0 < \mu_{1_a} < \mu_{1_b} < 1.5$. This dichotomy does not correspond to any obvious symmetry of the annealed partition function (2.3). Furthermore, there is no simple analytical expression for ΔJ_{c_1}. In contrast to the $n=2$ model, the diagonal solutions for the $n=3$ model follow the same pattern for all p and do not change pattern if $p > 4$.

The stability analysis in Section 4.2 will show that only the solutions μ_{1_a} are stable.

The second nonsingular solution Q of the order parameter equations (4.9) and (4.13) is

$$Q = \begin{pmatrix} 1 & q_o & q_o \\ q_o & 1 & q_o \\ q_o & q_o & 1 \end{pmatrix} \tag{4.28}$$

Its order parameter q_o is a solution of

$$1 - q_o = \frac{3}{2z}\left[\frac{1+2z}{2} - \left(\frac{z}{\pi}\right)^{1/2} \frac{e^z}{\operatorname{erfi}(\sqrt{z})}\right] \tag{4.29}$$

with

$$z = \frac{9p(\Delta J)^2}{4} q_o^{p-1} \tag{4.30}$$

This is a transcendental equation which can be solved numerically.

A numerical study of Eq. (4.29) shows that no solutions exist for small ΔJ, i.e., at high temperatures T.

The case $p=2$ is contained in Eq. (4.25). For all odd $p > 2$ we find one $\Delta J_{c_2} > \Delta J_{c_1}$ such that for all $\Delta J > \Delta J_{c_2}$, i.e., for all very low temperatures $T < T_{c_2}$ we have two solutions $0 < q_{o_b} < q_{o_a} < 1$. For all even $p > 2$ we find two transition points ΔJ_{c_2} and ΔJ_{c_3}. If $\Delta J_{c_2} < \Delta J < \Delta J_{c_3}$, i.e., for all (intermediate) low temperatures $T_{c_2} > T > T_{c_3}$ we have two solutions $0 < q_{o_b} < q_{o_a} < 1$. Above ΔJ_{c_3}, i.e., for all very low temperatures $T < T_{c_3}$ we find two negative solutions $-0.5 < q_{o_d} < q_{o_c}$ in addition to the positive solutions q_{o_a} and q_{o_b}. Again, there are no simple analytical expressions for ΔJ_{c_2} and ΔJ_{c_3}.

Our stability analysis in Section 4.2 will show that all these solutions are unstable.

Finally, we list the four singular solutions which exist only at $T = 0$,

$$Q_{\pm} = \begin{pmatrix} 0 & \pm 3 & 0 \\ \pm 3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad Q_{\pm} = \begin{pmatrix} 1.5 & \pm 1.5 & 0 \\ \pm 1.5 & 1.5 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.31}$$

The Q_+ exist for even and odd p's, while the Q_- exist only when p is even. These solutions are understood modulo constant orthonormal similarity transformations which rearrange the diagonal elements. The stability analysis in Section 4.2 shows that these solutions are excluded as well.

4.1.3. Case of Three Distinct Eigenvalues $\lambda_1 \neq \lambda_2 \neq \lambda_3$.

In this case the order parameter equations (4.5) cannot be further simplified and have to be evaluated numerically. We have performed an extensive numerical investigation and found that no solutions Q with $\lambda_1 \neq \lambda_2 \neq \lambda_3$ exist.

4.2. Stationarity of the Free Energy

By arguments analogous to the ones used for the stability analysis of the $n = 2$ model we can restrict our stability analysis to fluctuations δ_{11}, δ_{22}, δ_{12}, δ_{13}, and δ_{23} of the order parameters q_{11}, q_{22}, q_{12}, q_{13}, and q_{23}, respectively. The order parameters $\lambda_{\alpha\beta}$ are again just auxiliary quantities and the order parameter q_{33} is constrained to $q_{33} = (3 - q_{11} - q_{22})$ by our normalization condition for n-vectors. Further, the matrix Q must be symmetric because of Eq. (2.5).

The free energy for the $n = 3$ model is obtained from Eqs. (2.7), (2.8), and (2.12) in terms of these parameters as

$$\begin{aligned} \frac{a}{kT} &= \lim_{N \to \infty} \left(-\frac{1}{N} \ln \langle Z_N \rangle \right) \\ &= -G^* \end{aligned}$$

$$
\begin{aligned}
&= \frac{(p-1)(\Delta J)^2}{4}\left[q_{11}^p + q_{22}^p + (3 - q_{11} - q_{22})^p\right] \\
&\quad + \frac{(p-1)(\Delta J)^2}{2}\left(|q_{12}|^p + |q_{13}|^p + |q_{23}|^p\right) \\
&\quad - \ln \int_{\|\mathbf{S}\| = \sqrt{3}} \exp\left\{\frac{p(\Delta J)^2}{4}\left[q_{11}^{p-1}(S^1)^2 + q_{22}^{p-1}(S^2)^2\right.\right. \\
&\quad \left. + (3 - q_{11} - q_{22})^{p-1}(S^3)^2\right] \\
&\quad \left. + \frac{p(\Delta J)^2}{2}\left(q_{12}^{p-1} S^1 S^2 + q_{13}^{p-1} S^1 S^3 + q_{23}^{p-1} S^2 S^3\right)\right\} d\mathbf{S} \qquad (4.32) \\
&= \frac{(p-1)(\Delta J)^2}{4}\left[q_{11}^p + q_{22}^p + (3 - q_{11} - q_{22})^p\right] \\
&\quad + \frac{(p-1)(\Delta J)^2}{2}\left(|q_{12}|^p + |q_{13}|^p + |q_{23}|^p\right) \\
&\quad - \ln \frac{\operatorname{erfi}\{[3p(\Delta J)^2/4](\lambda_\gamma - \lambda_\alpha)\}^{1/2}}{\{[3p(\Delta J)^2/4](\lambda_\gamma - \lambda_\alpha)\}^{1/2}} - \frac{3p(\Delta J)^2}{4}\lambda_\alpha - \ln 6\pi^{3/2} \qquad (4.33)
\end{aligned}
$$

In the evaluation of the integral in the last step we have exploited the fact that there are no solutions with $\lambda_1 \neq \lambda_2 \neq \lambda_3$ and we have made the coordinate transformation (2.15) and used Eqs. (2.18), (2.21), and (A27). The eigenvalues of $Q^{(p-1)}$ are denoted by $\lambda_\alpha = \lambda_\beta \neq \lambda_\gamma$ as in the previous section.

The free energy (4.32) will be stationary (stable) if it constitutes a local minimum with respect to fluctuations δ_{11}, δ_{22}, δ_{12}, δ_{13}, and δ_{23} about the solution points $q_{11}, \dots, q_{23}$. This has to be investigated for all the possible solutions Q of the order parameter equations which we obtained in the previous section.

4.2.1. Case $Q = \operatorname{diag}(1, 1, 1)$. The free energy in Eq. (4.32) will be stable if its Hessian matrix with respect to $q_{11}, \dots, q_{23}$ is positive definite. Evaluating the Hessian matrix

$$
H \equiv \begin{pmatrix}
\frac{\partial^2}{\partial q_{11}\,\partial q_{11}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{11}\,\partial q_{22}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{11}\,\partial q_{12}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{11}\,\partial q_{13}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{11}\,\partial q_{23}}\frac{a}{kT} \\
\frac{\partial^2}{\partial q_{11}\,\partial q_{22}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{22}\,\partial q_{22}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{22}\,\partial q_{12}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{22}\,\partial q_{13}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{22}\,\partial q_{23}}\frac{a}{kT} \\
\frac{\partial^2}{\partial q_{11}\,\partial q_{12}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{22}\,\partial q_{12}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{12}\,\partial q_{12}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{12}\,\partial q_{13}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{12}\,\partial q_{23}}\frac{a}{kT} \\
\frac{\partial^2}{\partial q_{11}\,\partial q_{13}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{22}\,\partial q_{13}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{12}\,\partial q_{13}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{13}\,\partial q_{13}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{13}\,\partial q_{23}}\frac{a}{kT} \\
\frac{\partial^2}{\partial q_{11}\,\partial q_{23}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{22}\,\partial q_{23}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{12}\,\partial q_{23}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{13}\,\partial q_{23}}\frac{a}{kT} & \frac{\partial^2}{\partial q_{23}\,\partial q_{23}}\frac{a}{kT}
\end{pmatrix} \qquad (4.34)
$$

at the solution point $q_{11}=q_{22}=1$, $q_{12}=q_{13}=q_{23}=0$ gives, after a considerable amount of algebra, and by using Eqs. (B5), (B8), and (B10) from Appendix B in the evaluation of the occurring integrals,

$$H=\begin{pmatrix} h_1 & h_2 & 0 & 0 & 0 \\ h_2 & h_1 & 0 & 0 & 0 \\ 0 & 0 & h_3 & 0 & 0 \\ 0 & 0 & 0 & h_3 & 0 \\ 0 & 0 & 0 & 0 & h_3 \end{pmatrix} \tag{4.35}$$

with

$$h_1=\frac{p(p-1)(\Delta J)^2}{2}\left[1-\frac{3p(p-1)(\Delta J)^2}{10}\right] \tag{4.36}$$

$$h_2=h_1/2 \tag{4.37}$$

$$h_3=\begin{cases}(\Delta J)^2\,[1-\frac{3}{5}(\Delta J)^2] & (p=2)\\ 0 & (p>2)\end{cases} \tag{4.38}$$

The two eigenvalues of the h_1, h_2 submatrix of the Hessian (4.35) are

$$\begin{aligned}\xi_1&=\frac{p(p-1)(\Delta J)^2}{4}\left[1-\frac{3p(p-1)(\Delta J)^2}{10}\right]\\ \xi_2&=3\xi_1\end{aligned} \tag{4.39}$$

These equations show that for $p=2$ the high-temperature solution is stable if $\Delta J<\{10/[3p(p-1)]\}^{1/2}=(5/3)^{1/2}$ and unstable if $\Delta J>\{10/[3p(p-1)]\}^{1/2}$.

For $p>2$, the high-temperature solution will become unstable if $\Delta J>\{10/[3p(p-1)]\}^{1/2}$. If $\Delta J<\{10/[3p(p-1)]\}^{1/2}$, however, the Hessian is positive semidefinite and we have to look for higher-order fluctuations, as for the $n=2$ model. By using the relation (3.45), we find the following expansion for the free energy (4.32) about the solution point $q_{11}=q_{22}=1$, $q_{12}=q_{13}=q_{23}=0$:

$$\begin{aligned}\frac{a}{kT}=&-\frac{3(\Delta J)^2}{4}-\ln 12\pi\\ &+(\delta_{11},\delta_{22})\cdot\begin{pmatrix} h_1 & h_2\\ h_2 & h_1\end{pmatrix}\cdot\begin{pmatrix}\delta_{11}\\ \delta_{22}\end{pmatrix}+\frac{(p-1)(\Delta J)^2}{2}(|\delta_{12}|^p+|\delta_{13}|^p+|\delta_{23}|^p)\\ &+O(\delta_{11}\delta_{22}^2)+O(\delta_{11}^2\delta_{22})+O(\delta_{11}^3)+O(\delta_{22}^3)\\ &+O(\delta_{12}^{2p-2})+O(\delta_{13}^{2p-2})+O(\delta_{23}^{2p-2})\end{aligned} \tag{4.40}$$

Equation (4.39) tells us that the h_1, h_2 matrix is positive definite if $\Delta J < \{10/[3p(p-1)]\}^{1/2}$ and indefinite if $\Delta J > \{10/[3p(p-1)]\}^{1/2}$.

Thus, for all $p > 2$ the free energy corresponding to the high-temperature solution is stationary for $\Delta J < \{10/[3p(p-1)]\}^{1/2}$ due to pth-order fluctuations in δ_{12}, δ_{13}, and δ_{23}, and becomes unstable if $\Delta J > \{10/[3p(p-1)]\}^{1/2}$.

4.2.2. $p = 2$ and $Q = O^T \cdot \mathrm{diag}(\mu_1, \mu_1, 3 - 2\mu_1) \cdot O$. For $p = 2$, the free energy from Eq. (4.32) becomes

$$\frac{a}{kT} = \frac{(\Delta J)^2}{4} \sum_{\alpha,\beta=1}^{3} q_{\alpha\beta}^2 - \ln \int_{\|\mathbf{S}\| = \sqrt{3}} \exp\left[\frac{(\Delta J)^2}{2} \mathbf{S}^T Q \mathbf{S}\right] d\mathbf{S} \tag{4.41}$$

$\sum_{\alpha,\beta=1}^{3} q_{\alpha\beta}^2$ represents the square of the Euclidean matrix norm of the matrix Q. It is easy to show that the Euclidean matrix norm is invariant under orthonormal similarity transformations. Further, the integral on the right-hand side of Eq. (4.41) is invariant under orthonormal similarity transformations, as we saw in Section 2. Thus, for $p = 2$, a/kT is invariant under orthonormal similarity transformations of the matrix Q. This is analogous to the $n = 2$, $p = 2$ model, and corresponds to the invariance of the annealed partition function (2.3) under orthonormal transformations of the spin vectors $\mathbf{S}_i$ when $p = 2$. Finally, we have the constraint $\mu_3 = 3 - \mu_1 - \mu_2$ resulting from our normalization condition for n-vectors.

The stability of a/kT is therefore completely determined by the fluctuations of the eigenvalues μ_1 and μ_2 about their equilibrium values $\mu_1 = \mu_2$.

Rewriting Eq. (4.41) in terms of the eigenvalues μ_1 and μ_2 gives

$$\begin{aligned}\frac{a}{kT} &= \frac{(\Delta J)^2}{4} [\mu_1^2 + \mu_2^2 + (3 - \mu_1 - \mu_2)^2] \\ &\quad - \ln \int_{\|\mathbf{S}\| = \sqrt{3}} \exp\left\{\frac{(\Delta J)^2}{2} [\mu_1 (S^1)^2 + \mu_2 (S^2)^2 \right. \\ &\quad \left. + (3 - \mu_1 - \mu_2)(S^3)^2]\right\} d\mathbf{S}\end{aligned} \tag{4.42}$$

This free energy will be a minimum with respect to fluctuations δ_1 and δ_2 about the equilibrium values $\mu_1 = \mu_2$ if its Hessian matrix with respect to μ_1 and μ_2 is positive definite.

We can evaluate this Hessian at the equilibrium values $\mu_1 = \mu_2$ by using the relations (B5), (B8), and (B10) from Appendix B. After a considerable amount of algebra we obtain the following two eigenvalues for the Hessian of Eq. (4.42):

$$\xi_1 = \frac{(\Delta J)^2}{2}\left\{3 - (\Delta J)^2\left[\frac{6}{5}\frac{M[\frac{1}{2},\frac{7}{2},z] - M[\frac{3}{2},\frac{7}{2},z]}{M[\frac{1}{2},\frac{3}{2},z]} + \frac{9}{5}\frac{M[\frac{5}{2},\frac{7}{2},z]}{M[\frac{1}{2},\frac{3}{2},z]}\right.\right.$$

$$\left.\left. - \frac{M[\frac{1}{2},\frac{5}{2},z]^2}{M[\frac{1}{2},\frac{3}{2},z]^2} - \frac{M[\frac{3}{2},\frac{5}{2},z]^2}{M[\frac{1}{2},\frac{3}{2},z]^2} + 2\frac{M[\frac{1}{2},\frac{5}{2},z]\,M[\frac{3}{2},\frac{5}{2},z]}{M[\frac{1}{2},\frac{3}{2},z]^2}\right]\right\}$$

$$\xi_2 = \frac{(\Delta J)^2}{2}\left[1 - \frac{3}{5}(\Delta J)^2\frac{M[\frac{1}{2},\frac{7}{2},z]}{M[\frac{1}{2},\frac{3}{2},z]}\right] \tag{4.43}$$

with

$$z \equiv \frac{3p(\Delta J)^2}{4}\left[(3-2\mu_1)^{p-1} - \mu_1^{p-1}\right] \tag{4.44}$$

and where $M[\alpha, \beta, z]$ represents Kummer's hypergeometric function ${}_1F_1$ defined in Appendix A, Eq. (A8).

A numerical plot of ξ_1 and ξ_2 for the two possible solutions μ_{1_a} and μ_{1_b} obtained in Section 4.1 shows that the Hessian is indefinite for μ_{1_b} at all temperatures. In the case of μ_{1_a}, however, it is positive definite for all $\Delta J > \Delta J_{c_1} \approx 1.22306$.

Combined with our remarks above, this means that the low-temperature solution (4.25) for $p=2$ is unstable if $\mu_1 = \mu_{1_b}$. For $\mu_1 = \mu_{1_a}$, however, it is stable for all $\Delta J > \Delta J_{c_1}$, i.e., for all $T < T_{c_1}$. Since $\Delta J_{c_1} < \{10/[3p(p-1)]\}^{1/2}$, both the high-temperature solution and the solution (4.25) with μ_{1_a} are stable for $\Delta J_{c_1} < \Delta J < \{10/[3p(p-1)]\}^{1/2}$. We then have to find the dominant saddle point by comparing the respective free energies as for the $n=2$ model.

4.2.3. $p > 2$ and $Q = P^T \cdot \mathrm{diag}(\mu_1, \mu_1, 3-2\mu_1) \cdot P$. By using the relations (B5), (B8), and (B10) from Appendix B, we can evaluate the Hessian (4.34) for the free energy (4.32). After a considerable amount of algebra we obtain the Hessian

$$H = \begin{pmatrix} h_1 & h_2 & 0 & 0 & 0 \\ h_2 & h_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.45}$$

with the two eigenvalues of the h_1, h_2 submatrix given by

$$\xi_1 = \frac{p(p-1)^2 (\Delta J)^2}{4} [q_{11}^{p-2} + 2(3-2q_{11})^{p-2}]$$
$$- \frac{p^2(p-1)^2 (\Delta J)^4}{16} \left[q_{11}^{2(p-2)} \left(\frac{12}{5} \frac{M[\frac{1}{2}, \frac{7}{2}, z]}{M[\frac{1}{2}, \frac{3}{2}, z]} - 2 \frac{M[\frac{1}{2}, \frac{5}{2}, z]^2}{M[\frac{1}{2}, \frac{3}{2}, z]^2} \right) \right.$$
$$+ (3-2q_{11})^{2(p-2)} \left(\frac{18}{5} \frac{M[\frac{5}{2}, \frac{7}{2}, z]}{M[\frac{1}{2}, \frac{3}{2}, z]} - 2 \frac{M[\frac{3}{2}, \frac{5}{2}, z]^2}{M[\frac{1}{2}, \frac{3}{2}, z]^2} \right)$$
$$\left. - 2q_{11}^{p-2}(3-2q_{11})^{p-2} \left(\frac{6}{5} \frac{M[\frac{3}{2}, \frac{7}{2}, z]}{M[\frac{1}{2}, \frac{3}{2}, z]} - 2 \frac{M[\frac{1}{2}, \frac{5}{2}, z] \, M[\frac{3}{2}, \frac{5}{2}, z]}{M[\frac{1}{2}, \frac{3}{2}, z]} \right) \right]$$
$$- \frac{p(p-1)(p-2)(\Delta J)^2}{4} \left[q_{11}^{p-3} \frac{M[\frac{1}{2}, \frac{5}{2}, z]}{M[\frac{1}{2}, \frac{3}{2}, z]} \right.$$
$$\left. + 2(3-2q_{11})^{p-3} \frac{M[\frac{3}{2}, \frac{5}{2}, z]}{M[\frac{1}{2}, \frac{3}{2}, z]} \right]$$
$$\xi_2 = \frac{p(p-1)^2 (\Delta J)^2}{4} q_{11}^{p-2} - \frac{p^2(p-1)^2 (\Delta J)^4}{4} q_{11}^{2(p-2)} \frac{3}{10} \frac{M[\frac{1}{2}, \frac{7}{2}, z]}{M[\frac{1}{2}, \frac{3}{2}, z]}$$
$$- \frac{p(p-1)(p-2)(\Delta J)^2}{4} q_{11}^{p-3} \frac{M[\frac{1}{2}, \frac{5}{2}, z]}{M[\frac{1}{2}, \frac{3}{2}, z]} \tag{4.46}$$

The parameter z is given by Eq. (4.44). From the Hessian (4.45) and the relation (3.45) we then find the following expansion for the free energy (4.32) about the solution points $q_{11} = q_{22} \neq 1$, $q_{12} = q_{13} = q_{23} = 0$ when $p > 2$:

$$\frac{a}{kT} = -\ln 6\pi^{3/2} - \frac{3p(\Delta J)^2}{4} q_{11}^{p-1} + \frac{(p-1)(\Delta J)^2}{4} [2q_{11}^p + (3-2q_{11})^p]$$
$$- \ln \frac{\operatorname{erfi}\{[3p(\Delta J)^2/4][(3-2q_{11})^{p-1} - q_{11}^{p-1}]\}^{1/2}}{\{[3p(\Delta J)^2/4][(3-2q_{11})^{p-1} - q_{11}^{p-1}]\}^{1/2}}$$
$$+ (\delta_{11}, \delta_{22}) \cdot \begin{pmatrix} h_1 & h_2 \\ h_2 & h_1 \end{pmatrix} \cdot \begin{pmatrix} \delta_{11} \\ \delta_{22} \end{pmatrix} + \frac{(p-1)(\Delta J)^2}{2} (|\delta_{12}|^p + |\delta_{13}|^p + |\delta_{23}|^p)$$
$$+ O(\delta_{11}\delta_{22}^2) + O(\delta_{11}^2\delta_{22}) + O(\delta_{11}^3) + O(\delta_{22}^3)$$
$$+ O(\delta_{12}^{2p-2}) + O(\delta_{13}^{2p-2}) + O(\delta_{23}^{2p-2}) \tag{4.47}$$

Here we have used Eq. (4.33) in determining the zeroth-order term, and the eigenvalues ξ_1 and ξ_2 of the h_1, h_2 matrix are given in Eq. (4.46). Analogous expansions are obtained when $q_{11} = q_{33}$ or when $q_{22} = q_{33}$.

We have plotted ξ_1 and ξ_2 for the two possible solutions μ_{1_a} and μ_{1_b} obtained in Section 4.1 for several values of $p > 2$. In all cases we find that the Hessian is indefinite for μ_{1_b} at all temperatures. For μ_{1_a}, however, it is positive definite for all $\Delta J > \Delta J_{c_1}$.

This means that the low-temperature solutions $Q = P^T \cdot \mathrm{diag}(\mu_{1_b}, \mu_{1_b}, 3-2\mu_{1_b}) \cdot P$ are unstable while the solutions $Q = P^T \cdot \mathrm{diag}(\mu_{1_a}, \mu_{1_a}, 3-2\mu_{1_a}) \cdot P$ are stable for all $\Delta J > \Delta J_{c_1}$, i.e., for all $T < T_{c_1}$. Since $\Delta J_{c_1} < \{10/[3p(p-1)]\}^{1/2}$, both the high-temperature solution $Q = \mathrm{diag}(1, 1, 1)$ and the solution $Q = P^T \cdot \mathrm{diag}(\mu_{1_a}, \mu_{1_a}, 3-2\mu_{1_a}) \cdot P$ are stable for $\Delta J_{c_1} < \Delta J < \{10/[3p(p-1)]\}^{1/2}$. We then have to find the dominant saddle point by comparing the respective free energies as for the case $p = 2$.

4.2.4. $p > 2$ and $q_{11} = q_{22} = q_{33} = 1$, $q_{12} = q_{13} = q_{23} = q_0$. The Hessian matrix (4.34) for the free energy (4.32) can in this case be evaluated by using the relations (C13), (C19), and (C28) from Appendix C. After a considerable amount of algebra one finds the Hessian

$$H = \begin{pmatrix} a & a/2 & c & 0 & -c \\ a/2 & a & c & -c & 0 \\ c & c & b & d & d \\ 0 & -c & d & b & d \\ -c & 0 & d & d & b \end{pmatrix} \tag{4.48}$$

with

$$\begin{aligned}
a &= \frac{p(p-1)(\Delta J)^2}{2} - \frac{p^2(p-1)^2\,(\Delta J)^4}{20}\left[2\frac{M[\frac{3}{2},\frac{7}{2},x]}{M[\frac{1}{2},\frac{3}{2},x]} + \frac{M[\frac{1}{2},\frac{7}{2},x]}{M[\frac{1}{2},\frac{3}{2},x]}\right] \\
b &= \frac{p(p-1)^2\,(\Delta J)^2}{2}\,q_o^{p-2} \\
&\quad + \frac{p^2(p-1)^2\,(\Delta J)^4}{4}\,q_o^{2(p-2)}\left[\frac{1}{9}\left(\frac{M[\frac{3}{2},\frac{5}{2},x]}{M[\frac{1}{2},\frac{3}{2},x]} - \frac{M[\frac{1}{2},\frac{5}{2},x]}{M[\frac{1}{2},\frac{3}{2},x]}\right)^2\right. \\
&\quad \left.- \frac{1}{5}\left(\frac{M[\frac{5}{2},\frac{7}{2},x]}{M[\frac{1}{2},\frac{3}{2},x]} + 2\frac{M[\frac{1}{2},\frac{7}{2},x]}{M[\frac{1}{2},\frac{3}{2},x]}\right)\right] \\
&\quad - \frac{p(p-1)(p-2)(\Delta J)^2}{2}\,q_o^{p-3}\left[\frac{M[\frac{3}{2},\frac{5}{2},x]}{M[\frac{1}{2},\frac{3}{2},x]} - \frac{M[\frac{1}{2},\frac{5}{2},x]}{M[\frac{1}{2},\frac{3}{2},x]}\right] \\
c &= -\frac{p^2(p-1)^2\,(\Delta J)^2}{20}\left[\frac{M[\frac{3}{2},\frac{7}{2},x]}{M[\frac{1}{2},\frac{3}{2},x]} - \frac{M[\frac{1}{2},\frac{7}{2},x]}{M[\frac{1}{2},\frac{3}{2},x]}\right] q_o^{p-2} \\
d &= \frac{p^2(p-1)^2\,(\Delta J)^4}{4}\,q_o^{2(p-2)}\left[\frac{1}{9}\left(\frac{M[\frac{3}{2},\frac{5}{2},x]}{M[\frac{1}{2},\frac{3}{2},x]} - \frac{M[\frac{1}{2},\frac{5}{2},x]}{M[\frac{1}{2},\frac{3}{2},x]}\right)^2\right. \\
&\quad \left.- \frac{1}{5}\left(\frac{M[\frac{5}{2},\frac{7}{2},x]}{M[\frac{1}{2},\frac{3}{2},x]} - \frac{M[\frac{3}{2},\frac{7}{2},x]}{M[\frac{1}{2},\frac{3}{2},x]}\right)\right]
\end{aligned} \tag{4.49}$$

and where we have defined

$$x \equiv \frac{9p(\Delta J)^2}{4} q_o^{p-1} \tag{4.50}$$

The eigenvalues of this matrix can be found by following a de Almeida and Thouless procedure—see the appendix in ref. 8. We find

$$\begin{aligned}
\xi_1 &= b + 2d \\
\xi_{2,3} &= \tfrac{1}{4}\{3a + 2b - 2d \pm [(3a - 2b + 2d)^2 + 48c^2]^{1/2}\} \\
\xi_{4,5} &= \tfrac{1}{4}\{a + 2b - 2d \pm [(a - 2b + 2d)^2 + 16c^2]^{1/2}\}
\end{aligned} \tag{4.51}$$

where a, b, c, and d have been defined in Eq. (4.49).

A numerical plot of the eigenvalue ξ_1 for the solutions q_{o_a}, q_{o_b}, q_{o_c}, and q_{o_d} obtained in the previous section shows that this eigenvalue is always negative. We have confirmed this for several values of p.

Thus, the Hessian cannot be positive definite or positive semidefinite and the free energy becomes unstable. There are no stable nondiagonal solutions when $p > 2$.

4.3. Summary

The free energy per spin for the $n = 3$ model was derived in Eq. (4.33) in the thermodynamic limit $N \to \infty$ by the method of steepest descent from Eq. (2.6). It will only hold for stable solutions of the order parameter equations. As for the $n = 2$ model, if we find more than one stable solution at a certain temperature T, the solution which yields the lowest free energy (4.33) will constitute the true equilibrium configuration.

For all p we found that between the temperatures T_{c_1} and T_s, corresponding to $\Delta J_s \equiv \{10/[3p(p-1)]\}^{1/2}$, both the high-temperature solution $Q = \mathrm{diag}(1, 1, 1)$ and the "antisymmetric" diagonal solution $Q = P^T \cdot \mathrm{diag}(\mu_{1_a}, \mu_{1_a}, 3 - 2\mu_{1_a}) \cdot P$ are stable. A numerical comparison of the free energy (4.33) for these two solutions for various p shows that in all cases we find exactly one ΔJ_c with $\Delta J_{c_1} < \Delta J_c < \Delta J_s$. For $\Delta J < \Delta J_c$, i.e., for $T > T_c$, the high-temperature free energy remains the lower free energy, whereas for $\Delta J > \Delta J_c$, i.e., for $T < T_c$, the "antisymmetric" free energy has a lower value.

We further find that the singular $T = 0$ solutions from Eq. (4.31) all yield a higher free energy (4.33) than the $T \to 0$ limit of the "antisymmetric" configuration

$$P^T \cdot \mathrm{diag}(\mu_{1_a}, \mu_{1_a}, 3 - 2\mu_{1_a}) \cdot P = P^T \cdot \mathrm{diag}(0, 0, 3) \cdot P$$

Combined with the results from the previous sections, this then allows us to describe the various cases for the $n=3$ model as follows.

4.3.1. Case $p=2$. The model can be described by one order parameter μ_1, and we have one phase transition at $\Delta J_c \approx 1.23037$. There is no simple analytic expression for ΔJ_c.

For $\Delta J > \Delta J_c$, i.e., $T < T_c$, the order parameter μ_1 is the solution $0 < \mu_{1_a} < 0.75$ of Eq. (4.26). For $\Delta J < \Delta J_c$, i.e., $T > T_c$, μ_1 is identically equal to 1.

The order parameter matrix Q is given by $Q = O^T \cdot \text{diag}(\mu_1, \mu_1, 3-2\mu_1) \cdot O$ with O an arbitrary orthonormal matrix. The free energy per spin is given in general by Eq. (4.33), or in particular by the zeroth-order term of Eq. (4.40).

Thus, for $T > T_c$ we have the high-temperature solution $Q = \text{diag}(1, 1, 1)$.

At $T = T_c$ we have a phase transition. It is a first-order phase transition since $-\partial a/\partial T$ has a discontinuity at $T = T_c$. Furthermore, the spin configuration $(q_{\alpha\beta})$ has a jump discontinuity at T_c. This is in contrast to the $n=2$ model, where the spin configuration is continuous at T_c.

For $T < T_c$, we obtain a degenerate low-temperature continuum of states $Q = O^T \cdot \text{diag}(\mu_{1_a}, \mu_{1_a}, 3-2\mu_{1_a}) \cdot O$. The degeneracy results from the invariance of the annealed partition function (2.3) under orthonormal transformations (three-dimensional rotations) of the spin vectors $\mathbf{S}_i$ when $p=2$.

4.3.2. Case $p>2$. The models can be described by one order parameter μ_1, and we have one phase transition at ΔJ_c. There is no simple analytic expression for ΔJ_c. However, we know that $\Delta J_c < \{10/[3p(p-1)]\}^{1/2}$.

For $\Delta J > \Delta J_c$, i.e., $T < T_c$, the order parameter μ_1 is the solution $0 < \mu_{1_a} < 1 - \Delta$ (for some Δ) of Eq. (4.26). For $\Delta J < \Delta J_c$, i.e., $T > T_c$, μ_1 is identically equal to 1.

The order parameter matrix is given by $Q = P^T \cdot \text{diag}(\mu_1, \mu_1, 3-2\mu_1) \cdot P$, where P represents an arbitrary permutation of the diagonal elements of diag$(\cdots)$. The free energy per spin is given by Eq. (4.33).

Thus, as for $p=2$, we have the high-temperature solution $Q = \text{diag}(1, 1, 1)$ when $T > T_c$.

At $T = T_c$ we have a phase transition. As for $p=2$, it is a first-order phase transition with the spin configuration $(q_{\alpha\beta})$ displaying a jump discontinuity at T_c. This is in contrast to the $n=2$ model, where the spin configuration is continuous at T_c for $p=3$ and $p=4$.

For $T < T_c$, we obtain the diagonal solution $Q =$

$P^T \cdot \text{diag}(\mu_{1_a}, \mu_{1_a}, 3-2\mu_{1_a}) \cdot P$ with two identical diagonal elements. The symmetry in the vector components S_i^1, S_i^2, and S_i^3 in the annealed partition function (2.3) is responsible for the permutation degeneracy.

Since $\Delta J_c < \{10/[3p(p-1)]\}^{1/2}$, we find $\Delta J_c \to 0$, i.e., $T_c \to \infty$ as $p \to \infty$. From Eq. (4.26) we further see that for finite ΔJ and $p \to \infty$ the only possible configurations are $Q = P^T \cdot \text{diag}(0, 0, 3) \cdot P$. This is also what we expect physically as the ordering element of interactions becomes dominant when $p \to \infty$.

5. MODEL FOR GENERAL n

In the previous two sections we found that for the $n=2$ and $n=3$ model only diagonal solutions Q of the order parameter equations (2.23)–(2.25) constitute stable solutions when $p > 2$. We found further that we have only one phase transition. Finally, the low-temperature solution Q has the same number of distinct eigenvalues and respective eigenvalue degeneracy at $T=0$ and at finite temperatures. It seems physically reasonable to expect these general features from the corresponding models for arbitrary n as well.

With the above four assumptions, we can then derive explicit forms of the order parameter equations (2.23)–(2.25) for general n.

5.1. The High-Temperature Solution

From our normalization condition for n-vectors and Eq. (2.5) we have

$$\sum_{\gamma=1}^{n} q_{\gamma\gamma} = n \tag{5.1}$$

and

$$q_{\gamma\gamma} \geqslant 0 \tag{5.2}$$

If we look for a solution Q of the order parameter equations (2.23)–(2.25) which has only one distinct eigenvalue, then the order parameter equation (2.24) already requires that

$$Q = I \tag{5.3}$$

By using the formula (A9) from Appendix A, it is easy to see that the order parameter equations (2.23) will be satisfied in this case as well.

Thus, $Q = I$ represents a legitimate solution of Eqs. (2.23)–(2.25). For $p > 2$, we can identify it as the high-temperature solution by the fact that the order parameter equations (2.23) yield $\mu_1 = \cdots = \mu_n = 1$ as $T \to \infty$ and by our initial assumption that all stable solutions Q must be diagonal when $p>2$. For $p=2$, we have shown at the end of Section 2 that it suffices to find diagonal solutions Q_d. The most general solution Q is then obtained from Eq. (2.28). Since, as for $p>2$, the order parameter equations (2.23) yield $\mu_1 = \cdots = \mu_n = 1$ as $T \to \infty$, $Q = I$ must also represent the high-temperature solution for $p=2$.

The free energy per spin in this case is obtained from Eqs. (2.7), (2.8), and (2.12) as

$$\begin{aligned}
\frac{a}{kT} &= \lim_{N \to \infty} \left(-\frac{1}{N} \ln\langle Z_N \rangle \right) \\
&= -G^* \\
&= \frac{(p-1)(\Delta J)^2}{4} \sum_{\alpha,\beta} |q_{\alpha\beta}|^p - \ln \int_{\|\mathbf{S}\| = \sqrt{n}} \exp\left[\frac{p(\Delta J)^2}{4} \mathbf{S}^T Q^{(p-1)} \mathbf{S} \right] d\mathbf{S} \\
&= -\frac{n(\Delta J)^2}{4} - \ln \frac{2\pi^{n/2} n^{(n-1)/2}}{\Gamma(n/2)}
\end{aligned} \tag{5.4}$$

where we have used Eq. (B5) in the evaluation of the integral in the last step.

5.2. The Low-Temperature Solution

Because of Eqs. (5.1) and (5.2), the expression $\sum_{\gamma=1}^{n} q_{\gamma\gamma}^p$ will assume its maximum value if one $q_{\alpha\alpha}$ equals n and all other $q_{\gamma\gamma}$ equal 0. From Eq. (2.4) for the annealed partition function we see that at $T=0$, i.e., as $\Delta J \to \infty$, the system will be in the ground state determined by the maximum value of $\sum_{\alpha,\beta=1}^{n} q_{\alpha\beta}^p$. Since we have assumed that Q is diagonal when $p>2$ and since we showed at the end of Section 2 that it suffices to find a diagonal solution Q_d for $p=2$, $\sum_{\alpha,\beta=1}^{n} q_{\alpha\beta}^p$ will then assume its maximum value for

$$Q_s = \operatorname{diag}(0, \ldots, 0, n) \qquad \text{at} \quad T=0 \tag{5.5}$$

The most general matrix Q is obtained by permuting the diagonal elements of the standard form Q_s when $p>2$ and by an arbitrary orthonormal similarity transformation when $p=2$; see Eq. (2.28).

Because we have further assumed that the number of distinct eigenvalues and their degeneracy is conserved at finite temperatures, we thus have the result that at low temperatures the order parameter matrix Q has two distinct eigenvalues, one with degeneracy $n-1$ and one nondegenerate.

This means that the most general low-temperature order parameter matrix Q must be of the form

$$\begin{aligned} Q &= P^T \cdot \mathrm{diag}[\mu_1, ..., \mu_1, n-(n-1)\mu_1] \cdot P, \qquad p>2 \\ Q &= O^T \cdot \mathrm{diag}[\mu_1, ..., \mu_1, n-(n-1)\mu_1] \cdot O, \qquad p=2 \end{aligned} \tag{5.6}$$

where the similarity transformation with P represents an arbitrary permutation of the diagonal elements of diag$[\cdots]$ and where O is an arbitrary orthonormal matrix.

Since we require two distinct eigenvalues, this equation tells us that $\mu_1 \neq 1$. Therefore, we can satisfy the order parameter equation (2.24) by choosing

$$a_0 \equiv \mu_1 - a_1 \mu_1^{p-1} \tag{5.7}$$

$$a_1 \equiv \frac{n(1-\mu_1)}{[n-(n-1)\mu_1]^{p-1} - \mu_1^{p-1}} \tag{5.8}$$

The integrals in the order parameter equations (2.23), on the other hand, can be evaluated by using the formula (A9) from Appendix A. With $\lambda_\alpha = \mu_1^{p-1}$ having $(n-1)$-fold degeneracy and $\lambda_\gamma = [n-(n-1)\mu_1]^{p-1}$ being nondegenerate, we find

$$\mu_1 = \frac{M[\frac{1}{2}, n/2+1, z]}{M[\frac{1}{2}, n/2, z]} \tag{5.9}$$

and

$$n-(n-1)\mu_1 = \frac{M[\frac{3}{2}, n/2+1, z]}{M[\frac{1}{2}, n/2, z]} \tag{5.10}$$

Here, we have used the identity

$$M[a, b, z] = e^z M[b-a, b, -z] \tag{5.11}$$

from ref. 9, Eq. (13.1.27), in Eq. (5.10), and we have defined the quantity

$$z \equiv \frac{np(\Delta J)^2}{4} \{[n-(n-1)\mu_1]^{p-1} - \mu_1^{p-1}\} \tag{5.12}$$

Equations (5.9) and (5.10) are not independent. By using the identity

$$(1+a-b)\, M[a, b, z] - aM[a+1, b, z] = -(b-1)\, M[a, b-1, z] \tag{5.13}$$

from ref. 9, Eq. (13.4.3), we find

$$(n-1)\cdot\mu_1 + [n-(n-1)\mu_1] = n \tag{5.14}$$

as expected from our normalization condition for n-vectors.

Thus, Eq. (5.9) is the only order parameter equation which has to be solved in order to find the low-temperature solution (5.6) for the general n model. This can easily be done numerically.

The free energy per spin for the low-temperature solution is obtained from Eqs. (2.7), (2.8), and (2.12) as

$$\begin{aligned}\frac{a}{kT} &= \lim_{N\to\infty}\left(-\frac{1}{N}\ln\langle Z_N\rangle\right)\\ &= -G^*\\ &= \frac{(p-1)(\Delta J)^2}{4}\sum_{\alpha,\beta}|q_{\alpha\beta}|^p - \ln\int_{\|\mathbf{S}\|=\sqrt{n}}\exp\left[\frac{p(\Delta J)^2}{4}\mathbf{S}^T Q^{(p-1)}\mathbf{S}\right]d\mathbf{S}\\ &= \frac{(p-1)(\Delta J)^2}{4}\{(n-1)\mu_1^p+[n-(n-1)\mu_1]^p\}\\ &\quad -\ln\frac{2\pi^{n/2}n^{(n-1)/2}}{\Gamma(n/2)}-\frac{np(\Delta J)^2}{4}\mu_1^{p-1}-\ln M\left[\frac{1}{2},\frac{n}{2},z\right]\end{aligned}\tag{5.15}$$

where z has been defined in Eq. (5.12) and where we have used Eq. (B5) in the evaluation of the integral in the last step.

In order to prove the stationarity of the free energies (5.4) and (5.15) with respect to fluctuations of the order parameters $q_{\alpha\beta}$ about their respective equilibrium configurations, one would have to follow the same procedure as for the $n=2$ and $n=3$ models. The phase transition points ΔJ_c are determined by equating the two free energies (5.4) and (5.15).

APPENDIX A

In this Appendix we evaluate the following two types of integrals:

$$\hat{g}(\sqrt{n}) \equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp[\lambda n]\prod_{\gamma=1}^{n}\left(\lambda-\frac{p(\Delta J)^2}{4}\lambda_\gamma\right)^{-1/2}d\lambda \tag{A1}$$

$$\hat{f}_\alpha(\sqrt{n}) \equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp[\lambda n]\left(\lambda-\frac{p(\Delta J)^2}{4}\lambda_\alpha\right)^{-1}\prod_{\gamma=1}^{n}\left(\lambda-\frac{p(\Delta J)^2}{4}\lambda_\gamma\right)^{-1/2}d\lambda \tag{A2}$$

for the special cases $n=2$ and $n=3$.

Case $n=2$. From the integral representation of the gamma function

$$\int_0^\infty e^{-\lambda x}e^{ax}x^{\tau-1}\,dx = \Gamma(\tau)(\lambda-a)^{-\tau} \tag{A3}$$

and the convolution theorem for Laplace transforms we get

$$\int_0^\infty e^{-\lambda x}\,dx \int_0^\infty e^{a(x-t)}(x-t)^{\tau_1-1}\,e^{bt}t^{\tau_2-1}\,dt$$

$$=\Gamma(\tau_1)\,\Gamma(\tau_2)(\lambda-a)^{-\tau_1}\,(\lambda-b)^{-\tau_2} \tag{A4}$$

Using an integral representation of Kummer's confluent hypergeometric function [see e.g., ref. 9, Eq. (13.2.1)]

$$M(\alpha,\beta,z)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)\,\Gamma(\alpha)}\int_0^1 e^{zt}t^{\alpha-1}(1-t)^{\beta-\alpha-1}\,dt \tag{A5}$$

one can easily show that

$$\int_0^x e^{a(x-t)}(x-t)^{\tau_1-1}\,e^{bt}t^{\tau_2-1}\,dt$$

$$=e^{ax}B(\tau_1,\tau_2)x^{\tau_1+\tau_2-1}M(\tau_2,\tau_1+\tau_2,(b-a)x) \tag{A6}$$

Here, $B(\tau_1, \tau_2)$ is the beta function defined as

$$B(\tau_1,\tau_2)\equiv\frac{\Gamma(\tau_1)\,\Gamma(\tau_2)}{\Gamma(\tau_1+\tau_2)} \tag{A7}$$

and Kummer's function is defined as

$$M(\alpha,\beta,z)\equiv 1+\frac{\alpha}{\beta}\frac{z}{1!}+\frac{\alpha(\alpha+1)}{\beta(\beta+1)}\frac{z^2}{2!}+\cdots \tag{A8}$$

By inserting Eq. (A6) into Eq. (A4) and using Laplace's inversion formula, we finally find

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}(\lambda-a)^{-\tau_1}\,(\lambda-b)^{-\tau_2}\,e^{\lambda x}\,d\lambda$$

$$=\frac{e^{ax}x^{\tau_1+\tau_2-1}}{\Gamma(\tau_1+\tau_2)}M(\tau_2,\tau_1+\tau_2,(b-a)x) \tag{A9}$$

Applying this formula to Eq. (A1) when $n=2$ then gives

$$\hat{g}(\sqrt{2})\equiv\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp[2\lambda]\left(\lambda-\frac{p(\Delta J)^2}{4}\lambda_1\right)^{-1/2}\left(\lambda-\frac{p(\Delta J)^2}{4}\lambda_2\right)^{-1/2}d\lambda$$

$$=\exp\left[2\frac{p(\Delta J)^2}{4}\lambda_1\right]M\left(\frac{1}{2},1,2\frac{p(\Delta J)^2}{4}(\lambda_2-\lambda_1)\right)$$

$$=\exp\left[\frac{p(\Delta J)^2}{4}(\lambda_1+\lambda_2)\right]I_0\left(\frac{p(\Delta J)^2}{4}(\lambda_2-\lambda_1)\right) \tag{A10}$$

where in the last step we have used the identity

$$M(\tfrac{1}{2}, 1, 2z) = e^z I_0(z) \tag{A11}$$

[see, e.g., ref. 10, Eq. (7.11.2.10)] and where we have denoted modified Bessel functions of order n by I_n.

Applying formula (A9) to Eq. (A2), on the other hand, gives

$$\begin{aligned}\hat{f}_1(\sqrt{2}) &\equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp[2\lambda]\left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_1\right)^{-3/2}\left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_2\right)^{-1/2} d\lambda \\ &= 2\exp\left[2\frac{p(\Delta J)^2}{4}\lambda_1\right] M\left(\frac{1}{2}, 2, 2\frac{p(\Delta J)^2}{4}(\lambda_2 - \lambda_1)\right) \\ &= 2\exp\left[\frac{p(\Delta J)^2}{4}(\lambda_1 + \lambda_2)\right]\left[I_0\left(\frac{p(\Delta J)^2}{4}(\lambda_2 - \lambda_1)\right)\right. \\ &\quad \left. - I_1\left(\frac{p(\Delta J)^2}{4}(\lambda_2 - \lambda_1)\right)\right]\end{aligned} \tag{A12}$$

where in the last step we have used the identity

$$M(\tfrac{1}{2}, 2, 2z) = e^z[I_0(z) - I_1(z)] \tag{A13}$$

(see, e.g., ref. 10, Eq. (7.11.2.12)].

In the same fashion one finds

$$\begin{aligned}\hat{f}_2(\sqrt{2}) &= 2\exp\left[\frac{p(\Delta J)^2}{4}(\lambda_1 + \lambda_2)\right]\left[I_0\left(\frac{p(\Delta J)^2}{4}(\lambda_2 - \lambda_1)\right)\right. \\ &\quad \left. + I_1\left(\frac{p(\Delta J)^2}{4}(\lambda_2 - \lambda_1)\right)\right]\end{aligned} \tag{A14}$$

Case $n = 3$. From Eqs. (A9) and (A11) we find the Laplace inversion formula

$$\begin{aligned}&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp[\lambda x]\left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_\beta\right)^{-1/2}\left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_\gamma\right)^{-1/2} d\lambda \\ &\quad = \exp\left[\frac{p(\Delta J)^2}{8}(\lambda_\beta + \lambda_\gamma)x\right] I_0\left(\frac{p(\Delta J)^2}{8}(\lambda_\beta - \lambda_\gamma)x\right)\end{aligned} \tag{A15}$$

From ref. 11, Eq. (2.3.5), we further have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp[\lambda x]\left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_\alpha\right)^{-k-1/2} d\lambda \\ &\quad = \frac{2^{2k}k!}{\sqrt{\pi}\,(2k)!}\exp\left[\frac{p(\Delta J)^2}{4}\lambda_\alpha x\right] x^{k-1/2}\end{aligned} \tag{A16}$$

By using the convolution theorem for Laplace transforms, we can combine Eqs. (A15) and (A16) to get

$$\hat{f}_\alpha(\sqrt{3}) \equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp[3\lambda]\left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_\alpha\right)^{-3/2} \times\left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_\beta\right)^{-1/2}\left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_\gamma\right)^{-1/2} d\lambda$$
$$= \frac{2}{\sqrt{\pi}}\int_0^3 \exp\left[\frac{p(\Delta J)^2}{4}\lambda_\alpha(3-\tau)\right](3-\tau)^{1/2} \times \exp\left[\frac{p(\Delta J)^2}{8}(\lambda_\beta+\lambda_\gamma)\tau\right] I_0\left(\frac{p(\Delta J)^2}{8}\tau\right) d\tau$$
$$= 6c_\alpha \int_0^1 \exp[-a_\alpha u](1-u)^{1/2} I_0(b_\alpha u)\, du \tag{A17}$$

where we have made the substitution $\tau = 3u$ in the last step and where we have defined the quantities

$$a_\alpha \equiv \frac{3p(\Delta J)^2}{8}(2\lambda_\alpha - \lambda_\beta - \lambda_\gamma) \tag{A18}$$

$$b_\alpha \equiv \frac{3p(\Delta J)^2}{8}(\lambda_\beta - \lambda_\gamma) \tag{A19}$$

$$c_\alpha \equiv \left(\frac{3}{\pi}\right)^{1/2} \exp\left[\frac{3p(\Delta J)^2}{4}\lambda_\alpha\right] \tag{A20}$$

$\hat{g}(\sqrt{3})$ is evaluated by following the same procedure as for $\hat{f}_\alpha(\sqrt{3})$. We find

$$\hat{g}(\sqrt{3}) \equiv \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \exp[3\lambda]\left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_\alpha\right)^{-1/2} \times\left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_\beta\right)^{-1/2}\left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_\gamma\right)^{-1/2} d\lambda$$
$$= c_\alpha \int_0^1 \exp[-a_\alpha u](1-u)^{-1/2} I_0(b_\alpha u)\, du \tag{A21}$$

where a_α, b_α, and c_α have been defined in Eqs. (A18)–(A20). We note that the last integral in Eq. (A21) is only apparently dependent on α.

In the case that two of the eigenvalues λ_1, λ_2, and λ_3 are equal, the integrals in Eqs. (A17) and (A21) can be evaluated even further.

Let us assume that $\lambda_\alpha = \lambda_\beta \neq \lambda_\gamma$. By using Eq. (A9) and the definition of $\hat{f}_\alpha$, we then find

$$\hat{f}_\alpha(\sqrt{3}) \equiv \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp[3\lambda] \left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_\alpha\right)^{-2} \left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_\gamma\right)^{-1/2} d\lambda$$

$$= 4c_\alpha M\left[\frac{1}{2}, \frac{5}{2}, \frac{3p(\Delta J)^2}{4}(\lambda_\gamma - \lambda_\alpha)\right]$$

$$= c_\alpha \frac{3}{z}\left[\frac{1+2z}{2}\left(\frac{\pi}{z}\right)^{1/2} \operatorname{erfi}(\sqrt{z}) - e^z\right] \tag{A22}$$

where $\operatorname{erfi}(z)$ denotes the error function of an imaginary argument

$$\operatorname{erfi}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z \exp[t^2]\, dt \tag{A23}$$

and where we have defined the quantity

$$z \equiv \frac{3p(\Delta J)^2}{4}(\lambda_\gamma - \lambda_\alpha) \tag{A24}$$

In the last step of Eq. (A22) we have used the identity

$$M\left[\frac{1}{2}, \frac{5}{2}, z\right] = \frac{3}{4z}\left[\frac{1+2z}{2}\left(\frac{\pi}{z}\right)^{1/2} \operatorname{erfi}(\sqrt{z}) - e^z\right] \tag{A25}$$

[see, e.g., ref. 10, Eq. (7.11.2.13)].

In the same fashion we evaluate $\hat{f}_\gamma$ and $\hat{g}$,

$$\hat{f}_\gamma(\sqrt{3}) \equiv \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp[3\lambda] \left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_\alpha\right)^{-1} \left(\lambda - \frac{p(\Delta J)^2}{4}\lambda_\gamma\right)^{-3/2} d\lambda$$

$$= 4c_\alpha M\left[\frac{3}{2}, \frac{5}{2}, \frac{3p(\Delta J)^2}{4}(\lambda_\gamma - \lambda_\alpha)\right]$$

$$= c_\alpha \frac{6}{z}\left[e^z - \frac{1}{2}\left(\frac{\pi}{z}\right)^{1/2} \operatorname{erfi}(\sqrt{z})\right] \tag{A26}$$

$$\hat{g}(\sqrt{3}) \equiv \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp[3\lambda] \left(\lambda - \frac{p(\Delta J)^2}{4} \lambda_\alpha\right)^{-1} \left(\lambda - \frac{p(\Delta J)^2}{4} \lambda_\gamma\right)^{-1/2} d\lambda$$

$$= 2c_\alpha M\left[\frac{1}{2}, \frac{3}{2}, \frac{3p(\Delta J)^2}{4} (\lambda_\gamma - \lambda_\alpha)\right]$$

$$= c_\alpha \left(\frac{\pi}{z}\right)^{1/2} \operatorname{erfi}(\sqrt{z}) \tag{A27}$$

where z has been defined in Eq. (A24) and where we have used the identities

$$M\left[\frac{3}{2}, \frac{5}{2}, z\right] = \frac{3}{2z}\left[e^z - \frac{1}{2}\left(\frac{\pi}{z}\right)^{1/2} \operatorname{erfi}(\sqrt{z})\right] \tag{A28}$$

$$M\left[\frac{1}{2}, \frac{3}{2}, z\right] = \frac{1}{2}\left(\frac{\pi}{z}\right)^{1/2} \operatorname{erfi}(\sqrt{z}) \tag{A29}$$

from ref. 10, Eqs. (7.11.2.29) and (7.11.2.11).

APPENDIX B

In this Appendix we evaluate the following three types of integrals:

$$g(\sqrt{n}) \equiv \int_{\|\mathbf{S}\| = \sqrt{n}} \exp\left[\frac{p(\Delta J)^2}{4} \mathbf{S}^T Q^{(p-1)} \mathbf{S}\right] d\mathbf{S} \tag{B1}$$

$$f_{\alpha\beta}(\sqrt{n}) \equiv \int_{\|\mathbf{S}\| = \sqrt{n}} S^\alpha S^\beta \exp\left[\frac{p(\Delta J)^2}{4} \mathbf{S}^T Q^{(p-1)} \mathbf{S}\right] d\mathbf{S} \tag{B2}$$

$$h_{\alpha\beta\gamma\delta}(\sqrt{n}) \equiv \int_{\|\mathbf{S}\| = \sqrt{n}} S^\alpha S^\beta S^\gamma S^\delta \exp\left[\frac{p(\Delta J)^2}{4} \mathbf{S}^T Q^{(p-1)} \mathbf{S}\right] d\mathbf{S} \tag{B3}$$

for the diagonal order parameter matrix

$$Q^{(p-1)} \equiv \operatorname{diag}(\underbrace{\lambda_1, ..., \lambda_1}_{k}, \underbrace{\lambda_2, ..., \lambda_2}_{n-k}) \tag{B4}$$

Case 1:

$$g(\sqrt{n}) \equiv \int_{\|\mathbf{S}\| = \sqrt{n}} \exp\left[\frac{p(\Delta J)^2}{4} \mathbf{S}^T Q^{(p-1)} \mathbf{S}\right] d\mathbf{S}$$

By Laplace-inverting Eq. (2.21) and using the formula (A9) derived in Appendix A, we find

$$g(\sqrt{n}) = X \cdot M\left[\frac{n-k}{2}, \frac{n}{2}, z\right] \tag{B5}$$

where we have defined the quantities

$$X \equiv \frac{2\pi^{n/2} n^{(n-1)/2}}{\Gamma(n/2)} \exp\left(\frac{np(\Delta J)^2}{4} \lambda_1\right) \tag{B6}$$

and

$$z \equiv \frac{np(\Delta J)^2}{4} (\lambda_2 - \lambda_1) \tag{B7}$$

Case 2:

$$f_{\alpha\beta}(\sqrt{n}) \equiv \int_{\|\mathbf{S}\| = \sqrt{n}} S^\alpha S^\beta \exp\left[\frac{p(\Delta J)^2}{4} \mathbf{S}^T Q^{(p-1)} \mathbf{S}\right] d\mathbf{S}$$

By Laplace-inverting Eq. (2.20) and using the formula (A9), we get

$$f_{\alpha\beta}(\sqrt{n}) = \delta_{\alpha\beta} X \cdot \begin{cases} M\left[\frac{n-k}{2}, \frac{n}{2}+1, z\right], & \alpha \leqslant k \\ M\left[\frac{n-k}{2}+1, \frac{n}{2}+1, z\right], & \alpha > k \end{cases} \tag{B8}$$

where $\delta_{\alpha\beta}$ denotes the Kronecker δ-symbol and where X and z have been defined in Eqs. (B6) and (B7).

Case 3:

$$h_{\alpha\beta\gamma\delta}(\sqrt{n}) \equiv \int_{\|\mathbf{S}\| = \sqrt{n}} S^\alpha S^\beta S^\gamma S^\delta \exp\left[\frac{p(\Delta J)^2}{4} \mathbf{S}^T Q^{(p-1)} \mathbf{S}\right] d\mathbf{S}$$

As before, in order to avoid the $\sqrt{n}$ constraint in the above integration, we form the Laplace transform

$$\begin{aligned} &\int_0^\infty \exp[-\lambda x] \frac{h_{\alpha\beta\gamma\delta}(\sqrt{x})}{2\sqrt{x}} dx \\ &\quad = \int_0^\infty e^{-\lambda r^2} h_{\alpha\beta\gamma\delta}(r)\, dr \\ &\quad = \int_{-\infty}^\infty S^\alpha S^\beta S^\gamma S^\delta \exp\left[-\sum_{\tau=1}^n \left(\lambda - \frac{p(\Delta J)^2}{4} \lambda_\gamma\right) (S^\tau)^2\right] dS^1 \cdots dS^n \end{aligned}$$

$$
= \begin{cases}
\dfrac{3\pi^{n/2}}{4}\left(\lambda - \dfrac{p(\Delta J)^2}{4}\lambda_1\right)^{-(k/2+2)}\left(\lambda - \dfrac{p(\Delta J)^2}{4}\lambda_2\right)^{-(n-k)/2}, \\
\qquad \alpha=\beta=\gamma=\delta, \quad \alpha\leqslant k \\
\dfrac{3\pi^{n/2}}{4}\left(\lambda - \dfrac{p(\Delta J)^2}{4}\lambda_1\right)^{-k/2}\left(\lambda - \dfrac{p(\Delta J)^2}{4}\lambda_2\right)^{-((n-k)/2+2)}, \\
\qquad \alpha=\beta=\gamma=\delta, \quad \alpha> k \\
\dfrac{\pi^{n/4}}{2}\left(\lambda - \dfrac{p(\Delta J)^2}{4}\lambda_1\right)^{-(k/2+2)}\left(\lambda - \dfrac{p(\Delta J)^2}{4}\lambda_2\right)^{-(n-k)/2}, \\
\qquad \alpha=\beta\neq\gamma=\delta, \quad \alpha,\gamma\leqslant k \\
\dfrac{\pi^{n/4}}{2}\left(\lambda - \dfrac{p(\Delta J)^2}{4}\lambda_1\right)^{-(k/2+1)}\left(\lambda - \dfrac{p(\Delta J)^2}{4}\lambda_2\right)^{-((n-k)/2+1)}, \\
\qquad \alpha=\beta\neq\gamma=\delta, \quad \alpha\leqslant k<\gamma \\
\dfrac{\pi^{n/4}}{2}\left(\lambda - \dfrac{p(\Delta J)^2}{4}\lambda_1\right)^{-k/2}\left(\lambda - \dfrac{p(\Delta J)^2}{4}\lambda_2\right)^{-((n-k)/2+2)}, \\
\qquad \alpha=\beta\neq\gamma=\delta, \quad \alpha,\gamma> k \\
0, \qquad \text{otherwise}
\end{cases} \tag{B9}
$$

The conditions in Eq. (B9) are hereby understood as modulo permutations of α, β, γ, and δ. By Laplace-inverting Eq. (B9) and using the formula (A9), we get

$$
h_{\alpha\beta\gamma\delta}(\sqrt{n}) = \frac{nX}{n+2}\cdot
\begin{cases}
3M\left[\dfrac{n-k}{2}, \dfrac{n}{2}+2, z\right], & \alpha=\beta=\gamma=\delta, \quad \alpha\leqslant k \\
3M\left[\dfrac{n-k}{2}+2, \dfrac{n}{2}+2, z\right], & \alpha=\beta=\gamma=\delta, \quad \alpha> k \\
M\left[\dfrac{n-k}{2}, \dfrac{n}{2}+2, z\right], & \alpha=\beta\neq\gamma=\delta, \quad \alpha,\gamma\leqslant k \\
M\left[\dfrac{n-k}{2}+1, \dfrac{n}{2}+2, z\right], & \alpha=\beta\neq\gamma=\delta, \quad \alpha\leqslant k<\gamma \\
M\left[\dfrac{n-k}{2}+2, \dfrac{n}{2}+2, z\right], & \alpha=\beta\neq\gamma=\delta, \quad \alpha,\gamma> k \\
0, & \text{otherwise}
\end{cases} \tag{B10}
$$

where X and z have been defined in Eqs. (B6) and (B7), respectively. The conditions in Eq. (B10) are again understood as modulo permutations of α, β, γ, and δ.

APPENDIX C

In this Appendix we evaluate the following three types of integrals:

$$g(\sqrt{n}) \equiv \int_{\|\mathbf{S}\| = \sqrt{n}} \exp\left[\frac{p(\Delta J)^2}{4} \mathbf{S}^T Q^{(p-1)} \mathbf{S}\right] d\mathbf{S} \tag{C1}$$

$$f_{\alpha\beta}(\sqrt{n}) \equiv \int_{\|\mathbf{S}\| = \sqrt{n}} S^\alpha S^\beta \exp\left[\frac{p(\Delta J)^2}{4} \mathbf{S}^T Q^{(p-1)} \mathbf{S}\right] d\mathbf{S} \tag{C2}$$

$$h_{\alpha\beta\gamma\delta}(\sqrt{n}) \equiv \int_{\|\mathbf{S}\| = \sqrt{n}} S^\alpha S^\beta S^\gamma S^\delta \exp\left[\frac{p(\Delta J)^2}{4} \mathbf{S}^T Q^{(p-1)} \mathbf{S}\right] d\mathbf{S} \tag{C3}$$

for the special symmetric order parameter matrix Q

$$q_{\alpha\beta} \equiv \begin{cases} q_d & \alpha = \beta \\ q_o & \alpha \neq \beta \end{cases} \tag{C4}$$

Case 1:

$$g(\sqrt{n}) \equiv \int_{\|\mathbf{S}\| = \sqrt{n}} \exp\left[\frac{p(\Delta J)^2}{4} \mathbf{S}^T Q^{(p-1)} \mathbf{S}\right] d\mathbf{S}$$

In order to avoid the $\sqrt{n}$ constraint in the above integration, we can form the Laplace transform

$$\begin{aligned} &\int_0^\infty \exp[-\lambda x] \frac{g(\sqrt{x})}{2\sqrt{x}} dx \\ &= \int_0^\infty \exp[-\lambda r^2]\, g(r)\, dr \\ &= \int_{-\infty}^\infty \exp\left[-\lambda \mathbf{S}^T\mathbf{S} + \frac{p(\Delta J)^2}{4} \mathbf{S}^T Q^{(p-1)} \mathbf{S}\right] d\mathbf{S} \\ &= \int_{-\infty}^\infty \exp[-\mathbf{S}^T A \mathbf{S}]\, d\mathbf{S} \end{aligned} \tag{C5}$$

where we have defined the matrix A as

$$A \equiv \begin{cases} \lambda - \dfrac{p(\Delta J)^2}{4} q_d^{p-1}, & \alpha = \beta \\[2ex] -\dfrac{p(\Delta J)^2}{4} q_o^{p-1}, & \alpha \neq \beta \end{cases} \tag{C6}$$

A is a cyclic matrix. Thus, its eigenvalues v'_i are determined by the nth roots of 1, ω_i $(i=1,\ldots,n)$,

$$v'_i = \lambda - \frac{p(\Delta J)^2}{4} q_d^{p-1} - \sum_{k=1}^{n-1} \frac{p(\Delta J)^2}{4} q_o^{p-1} \omega_i^k$$
$$= \begin{cases} \lambda - \dfrac{p(\Delta J)^2}{4} (q_d^{p-1} - q_o^{p-1}), & \omega_i \neq 1 \\ \lambda - \dfrac{p(\Delta J)^2}{4} (q_d^{p-1} + (n-1) q_o^{p-1}), & \omega_i = 1 \end{cases} \tag{C7}$$

Since A is symmetric, we can find an orthonormal matrix O which diagonalizes A. Using the definitions

$$\begin{aligned} v_1 &\equiv \lambda - \frac{p(\Delta J)^2}{4} [q_d^{p-1} + (n-1) q_o^{p-1}] \\ v_2 &\equiv \lambda - \frac{p(\Delta J)^2}{4} (q_d^{p-1} - q_o^{p-1}) \end{aligned} \tag{C8}$$

we can demand

$$D \equiv O^T A O = \operatorname{diag}(v_1, v_2, \ldots, v_2) \tag{C9}$$

The orthonormal matrix $O \equiv (o_{\alpha\beta})$ must then have the form

$$\begin{aligned} o_{\alpha 1} &= n^{-1/2} \\ o_{\alpha\beta} &= 0, & \beta > \alpha + 1, \quad \beta > 1 \\ o_{\alpha\beta} &= -(n - \beta + 1)^{1/2} (n - \beta + 2)^{-1/2}, & \beta = \alpha + 1, \quad \beta > 1 \\ o_{\alpha\beta} &= (n - \beta + 1)^{-1/2} (n - \beta + 2)^{-1/2}, & \beta < \alpha + 1, \quad \beta > 1 \end{aligned} \tag{C10}$$

By using the orthonormal coordinate transformation

$$\hat{\mathbf{S}} \equiv O^T \mathbf{S} \tag{C11}$$

we can now evaluate Eq. (C5),

$$\begin{aligned} &\int_{-\infty}^{\infty} \exp[-\mathbf{S}^T A \mathbf{S}]\, d\mathbf{S} \\ &\quad = \int_{-\infty}^{\infty} \exp\left[-v_1 (\hat{S}^1)^2 - \sum_{\beta=2}^{n} v_2 (\hat{S}^2)^2 \right] d\hat{S}^1 \cdots d\hat{S}^n \\ &\quad = \pi^{n/2} v_1^{-1/2} v_2^{-(n-1)/2} \end{aligned} \tag{C12}$$

Laplace inversion of Eq. (C5) and application of the formula (A9) derived in Appendix A to Eq. (C12) finally yields

$$g(\sqrt{n}) = Y \cdot M\left(\frac{n-1}{2}, \frac{n}{2}, z\right) \tag{C13}$$

where we have defined the quantities

$$Y \equiv \frac{2\pi^{n/2} n^{(n-1)/2}}{\Gamma(n/2)} \exp\left\{\frac{np(\Delta J)^2}{4} \left[q_d^{p-1} + (n-1) q_o^{p-1}\right]\right\} \tag{C14}$$

and

$$z \equiv -\frac{n^2 p(\Delta J)^2}{4} q_o^{p-1} \tag{C15}$$

Case 2:

$$f_{\alpha\beta}(\sqrt{n}) \equiv \int_{\|\mathbf{S}\| = \sqrt{n}} S^\alpha S^\beta \exp\left[\frac{p(\Delta J)^2}{4} \mathbf{S}^T Q^{(p-1)} \mathbf{S}\right] d\mathbf{S}$$

We evaluate $f_{\alpha\beta}(\sqrt{n})$ for the symmetric matrix given by Eq. (C4) and proceed analogously to the derivation of $g(\sqrt{n})$.

The $\sqrt{n}$ constraint can be avoided by first taking the Laplace transform

$$\int_0^\infty [\exp(-\lambda x)] \frac{f_{\alpha\beta}(\sqrt{x})}{2\sqrt{x}}\, dx = \int_0^\infty [\exp(-\lambda r^2)]\, f_{\alpha\beta}(r)\, dr$$

$$= \int_{-\infty}^{\infty} S^\alpha S^\beta \exp[-\mathbf{S}^T A \mathbf{S}]\, d\mathbf{S} \tag{C16}$$

where A is given by Eq. (C6).

The matrix A is then diagonalized by means of the orthonormal coordinate transformation (C11).

For $\alpha = \beta$ we get

$$\int_{-\infty}^{\infty} (S^\alpha)^2 \exp[-\mathbf{S}^T A \mathbf{S}]\, d\mathbf{S}$$

$$= \int_{-\infty}^{\infty} \left(\sum_{\beta=1}^{n} o_{\alpha\beta} \hat{S}^\beta\right)^2 \exp[-\hat{\mathbf{S}}^T D \hat{\mathbf{S}}]\, d\hat{\mathbf{S}}$$

$$= \int_{-\infty}^{\infty} \sum_{\beta=1}^{n} o_{\alpha\beta}^2 (\hat{S}^\beta)^2 \exp\left[-v_1(\hat{S}^1)^2 - \sum_{\beta=2}^{n} v_2(\hat{S}^\beta)^2 \right] d\hat{S}^1 \cdots d\hat{S}^n$$

$$= o_{\alpha 1}^2 \frac{\pi^{n/2}}{2} v_1^{-3/2} v_2^{-(n-1)/2} + \sum_{\beta=2}^{n} o_{\alpha\beta}^2 \frac{\pi^{n/2}}{2} v_1^{-1/2} v_2^{-(n+1)/2}$$

$$= \frac{\pi^{n/2}}{2n} [v_1^{-3/2} v_2^{-(n-1)/2} + (n-1) v_1^{-1/2} v_2^{-(n+1)/2}] \tag{C17}$$

where we have used Eq. (C10) in the last step.

For $\alpha \neq \beta$ we get

$$\int_{-\infty}^{\infty} S^\alpha S^\beta \exp[-\mathbf{S}^T A \mathbf{S}] \, d\mathbf{S}$$

$$= \int_{-\infty}^{\infty} \sum_{\gamma=1}^{n} o_{\alpha\gamma} o_{\beta\gamma} (\hat{S}^\gamma)^2 \exp\left[-v_1(\hat{S}^1)^2 - \sum_{\beta=2}^{n} v_2(\hat{S}^\beta)^2 \right] d\hat{S}^1 \cdots d\hat{S}^n$$

$$= o_{\alpha 1} o_{\beta 1} \frac{\pi^{n/2}}{2} v_1^{-3/2} v_2^{-(n-1)/2} + \sum_{\gamma=2}^{n} o_{\alpha\gamma} o_{\beta\gamma} \frac{\pi^{n/2}}{2} v_1^{-1/2} v_2^{-(n+1)/2}$$

$$= \frac{\pi^{n/2}}{2n} [v_1^{-3/2} v_2^{-(n-1)/2} - v_1^{-1/2} v_2^{-(n+1)/2}] \tag{C18}$$

$f_{\alpha\beta}(\sqrt{n})$ can now be determined by Laplace-inverting Eq. (C16) and applying the formula (A9) derived in Appendix A to Eqs. (C17) and (C18). We obtain

$$f_{\alpha\beta}(\sqrt{n}) = \frac{Y}{n} \cdot \begin{cases} M\left(\frac{n-1}{2}, \frac{n}{2}+1, z\right) - M\left(\frac{n+1}{2}, \frac{n}{2}+1, z\right), & \alpha \neq \beta \\ M\left(\frac{n-1}{2}, \frac{n}{2}+1, z\right) + (n-1)\, M\left(\frac{n+1}{2}, \frac{n}{2}+1, z\right), & \alpha = \beta \end{cases} \tag{C19}$$

where the quantities Y and z have been defined in Eqs. (C14) and (C15), respectively.

Case 3:

$$h_{\alpha\beta\gamma\delta}(\sqrt{n}) \equiv \int_{\|\mathbf{S}\| = \sqrt{n}} S^\alpha S^\beta S^\gamma S^\delta \exp\left[\frac{p(\Delta J)^2}{4} \mathbf{S}^T Q^{(p-1)} \mathbf{S} \right] d\mathbf{S}$$

Again, we evaluate $h_{\alpha\beta\gamma\delta}(\sqrt{n})$ for the symmetric matrix given by Eq. (C4) and proceed analogously to the derivation of $g(\sqrt{n})$.

The $\sqrt{n}$ constraint is circumvented by first taking the Laplace transform

$$\int_0^\infty [\exp(-\lambda x)] \frac{h_{\alpha\beta\gamma\delta}(\sqrt{x})}{2\sqrt{x}}\, dx = \int_0^\infty [\exp(-\lambda r^2)]\, h_{\alpha\beta\gamma\delta}(r)\, dr$$
$$= \int_{-\infty}^{\infty} S^\alpha S^\beta S^\gamma S^\delta \exp[-\mathbf{S}^T A \mathbf{S}]\, d\mathbf{S} \qquad \text{(C20)}$$

where A is given by Eq. (C6).

Let $P_{\rho\tau}$ be the $n \times n$ permutation matrix which swaps the ρth and τth components of $\mathbf{S}$,

$$P_{\rho\tau} \begin{pmatrix} S^1 \\ \vdots \\ S^\rho \\ \vdots \\ S^\tau \\ \vdots \\ S^n \end{pmatrix} = \begin{pmatrix} S^1 \\ \vdots \\ S^\tau \\ \vdots \\ S^\rho \\ \vdots \\ S^n \end{pmatrix} \qquad \text{(C21)}$$

We can easily show that because of the special symmetry of the matrix A [see Eq. (C6)] we have

$$P_{\rho\tau}^T A P_{\rho\tau} = A \qquad \text{for all } \rho, \tau \qquad \text{(C22)}$$

Thus, we can find coordinate transformations

$$\tilde{\mathbf{S}} \equiv P_{\rho_1\tau_1} \cdot \cdots \cdot P_{\rho_4\tau_4} \cdot \mathbf{S} \qquad \text{(C23)}$$

which permute the components of $\mathbf{S}$ in such a way that the last integral in Eq. (C20) reduces to one of the following five canonical forms:

$$\int_{-\infty}^{\infty} S^\alpha S^\beta S^\gamma S^\delta \exp[-\mathbf{S}^T A \mathbf{S}]\, d\mathbf{S}$$

$$= \begin{cases} \int_{-\infty}^{\infty} \tilde{S}^1 \tilde{S}^2 \tilde{S}^3 \tilde{S}^4 \exp[-\tilde{\mathbf{S}}^T A \tilde{\mathbf{S}}]\, d\tilde{\mathbf{S}}, & \alpha \neq \beta \neq \gamma \neq \delta \\ \int_{-\infty}^{\infty} (\tilde{S}^1)^2\, \tilde{S}^2 \tilde{S}^3 \exp[-\tilde{\mathbf{S}}^T A \tilde{\mathbf{S}}]\, d\tilde{\mathbf{S}}, & \alpha = \beta \neq \gamma \neq \delta \\ \int_{-\infty}^{\infty} (\tilde{S}^1)^2\, (\tilde{S}^2)^2 \exp[-\tilde{\mathbf{S}}^T A \tilde{\mathbf{S}}]\, d\tilde{\mathbf{S}}, & \alpha = \beta \neq \gamma = \delta \\ \int_{-\infty}^{\infty} (\tilde{S}^1)^3\, \tilde{S}^2 \exp[-\tilde{\mathbf{S}}^T A \tilde{\mathbf{S}}]\, d\tilde{\mathbf{S}}, & \alpha = \beta = \gamma \neq \delta \\ \int_{-\infty}^{\infty} (\tilde{S}^1)^4 \exp[-\tilde{\mathbf{S}}^T A \tilde{\mathbf{S}}]\, d\tilde{\mathbf{S}}, & \alpha = \beta = \gamma = \delta \end{cases} \qquad \text{(C24)}$$

Here we have used Eq. (C23) and the fact that $\det(P_{\rho\tau}) = 1$.

Using the orthonormal coordinate transformation

$$\hat{\mathbf{S}} \equiv O^T \tilde{\mathbf{S}} \tag{C25}$$

we can now evaluate Eq. (C24).

For $\alpha \neq \beta \neq \gamma \neq \delta$ we get

$$\int_{-\infty}^{\infty} S^\alpha S^\beta S^\gamma S^\delta \exp[-\mathbf{S}^T A \mathbf{S}]\, d\mathbf{S}$$

$$= \int_{-\infty}^{\infty} \tilde{S}^1 \tilde{S}^2 \tilde{S}^3 \tilde{S}^4 \exp[-\tilde{\mathbf{S}}^T A \tilde{\mathbf{S}}]\, d\tilde{\mathbf{S}}$$

$$= \int_{-\infty}^{\infty} \left(\sum_{\alpha=1}^{n} o_{1\alpha} \hat{S}^\alpha\right)\left(\sum_{\beta=1}^{n} o_{2\beta} \hat{S}^\beta\right)\left(\sum_{\gamma=1}^{n} o_{3\gamma} \hat{S}^\gamma\right)\left(\sum_{\delta=1}^{n} o_{4\delta} \hat{S}^\delta\right)$$

$$\times \exp\left[-v_1 (\hat{S}^1)^2 - \sum_{\beta=2}^{n} v_2 (\hat{S}^\beta)^2\right] d\hat{\mathbf{S}}$$

$$= \int_{-\infty}^{\infty} \left[\sum_{\alpha=1}^{n} o_{1\alpha} o_{2\alpha} o_{3\alpha} o_{4\alpha} (\hat{S}^\alpha)^4\right.$$

$$\left. + \sum_{\alpha \neq \beta} (o_{1\alpha} o_{2\alpha} o_{3\beta} o_{4\beta} + o_{1\alpha} o_{3\alpha} o_{2\beta} o_{4\beta} + o_{1\alpha} o_{4\alpha} o_{2\beta} o_{3\beta}) (\hat{S}^\alpha)^2 (\hat{S}^\beta)^2\right]$$

$$\times \exp\left[-v_1 (\hat{S}^1)^2 - \sum_{\beta=2}^{n} v_2 (\hat{S}^\beta)^2\right] d\hat{\mathbf{S}}$$

$$= \int_{-\infty}^{\infty} \left[\frac{1}{n^2} (\hat{S}^1)^4 - \frac{1}{(n-1)n^2} (\hat{S}^2)^4 - \frac{6}{n} (\hat{S}^1)^2 (\hat{S}^2)^2\right.$$

$$\left. + \frac{3}{n(n-1)} (\hat{S}^2)^2 (\hat{S}^3)^2\right] \exp\left[-v_1 (\hat{S}^1)^2 - \sum_{\beta=2}^{n} v_2 (\hat{S}^\beta)^2\right] d\hat{\mathbf{S}}$$

$$= \frac{\pi^{n/2}}{4n^2} (3 v_1^{-5/2} v_2^{-(n-1)/2} - 6 v_1^{-3/2} v_2^{-(n+1)/2} + 3 v_1^{-1/2} v_2^{-(n+3)/2}) \tag{C26}$$

where we have used the explicit form of the matrix O from Eq. (C10) and the symmetry of the integral in the second last step.

In a similar fashion one evaluates the remaining four cases of Eq. (C24). We find

$$\int_{-\infty}^{\infty} S^\alpha S^\beta S^\gamma S^\delta \exp[-\mathbf{S}^T A \mathbf{S}]\, d\mathbf{S}$$

$$= \begin{cases} \dfrac{\pi^{n/2}}{4n^2}\left(3v_1^{-5/2}v_2^{-(n-1)/2} + (n-6)v_1^{-3/2}v_2^{-(n+1)/2} \right. \\ \qquad \left. -(n-3)v_1^{-1/2}v_2^{-(n+3)/2}\right) & \alpha=\beta\neq\gamma\neq\delta \\ \dfrac{\pi^{n/2}}{4n^2}\left(3v_1^{-5/2}v_2^{-(n-1)/2} + 2(n-3)v_1^{-3/2}v_2^{-(n+1)/2} \right. \\ \qquad \left. +(n^2-2n+3)v_1^{-1/2}v_2^{-(n+3)/2}\right) & \alpha=\beta\neq\gamma=\delta \\ \dfrac{\pi^{n/2}}{4n^2}\left(3v_1^{-5/2}v_2^{-(n-1)/2} + 3(n-2)v_1^{-3/2}v_2^{-(n+1)/2} \right. \\ \qquad \left. -3(n-1)v_1^{-1/2}v_2^{-(n+3)/2}\right) & \alpha=\beta=\gamma\neq\delta \\ \dfrac{\pi^{n/2}}{4n^2}\left(3v_1^{-5/2}v_2^{-(n-1)/2} + 6(n-1)v_1^{-3/2}v_2^{-(n+1)/2} \right. \\ \qquad \left. +3(n-1)^2 v_1^{-1/2}v_2^{-(n+3)/2}\right) & \alpha=\beta=\gamma=\delta \end{cases} \tag{C27}$$

Now $h_{\alpha\beta\gamma\delta}(\sqrt{n})$ can be determined by Laplace-inverting Eq. (C20) and applying the formula (A9) derived in Appendix A to Eqs. (C26) and (C27). We obtain

$$h_{\alpha\beta\gamma\delta}(\sqrt{n}) = \frac{Y}{n(n+2)} \cdot \begin{cases} H[3, -6, 3, z], & \alpha\neq\beta\neq\gamma\neq\delta \\ H[3, n-6, -(n-3), z], & \alpha=\beta\neq\gamma\neq\delta \\ H[3, 2(n-3), n^2-2n+3, z], & \alpha=\beta\neq\gamma=\delta \\ H[3, 3(n-2), -3(n-1), z], & \alpha=\beta=\gamma\neq\delta \\ H[3, 6(n-1), 3(n-1)^2, z], & \alpha=\beta=\gamma=\delta \end{cases} \tag{C28}$$

where the quantities Y and z have been defined in Eqs. (C14) and (C15) and where we have introduced the function

$$H[a, b, c, z] \equiv aM\left(\frac{n-1}{2}, \frac{n}{2}+2, z\right) + bM\left(\frac{n+1}{2}, \frac{n}{2}+2, z\right) + cM\left(\frac{n+3}{2}, \frac{n}{2}+2, z\right) \tag{C29}$$

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